NOTES ON DRINFELD MODULAR FORMS

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1. Introduction

1.1. **Preface.** These are notes for the intensive course I gave at Tohoku university in the fall of 2024. I do not claim that anything in these notes is original: I just tried to explain the definition of Drinfeld modular forms and Hecke operators acting on them [Gos1, Gek1, Gek2], the description of Drinfeld cuspforms using harmonic cocycles on the Bruhat–Tits tree [Tei1], the necessary background on the tree [Ser] and on rigid analytic residue theorems [FvdP1]. In most parts I followed the normalization and exposition of [Böc], and I copied some arguments of [Pel] on the analysis around cusps, though any errors are my fault. It should be used at the reader's own risk.

1.2. **Convention.** We follow the convention in [Böc].

(1) We consider $V_\infty = K_\infty^2$ as the set of row vectors on which $GL_2(K_\infty)$ acts tautologically from the right and we define a left action $\circ$ of $GL_2(K_\infty)$ on V_∞ by

$$\gamma \circ (x, y) = (x, y)\gamma^{-1} = \left(\frac{dx - cy}{ad - bc}, \frac{-bx + ay}{ad - bc}\right), \quad \gamma \in GL_2(K_\infty)$$

(2) Put $\mathbb{P}^1(K_\infty) = (V_\infty \backslash \{(0,0)\})/K_\infty^\times$ and similarly for $\mathbb{P}^1(\mathbb{C}_\infty)$. The class of (x, y) is denoted by $(x : y)$. We consider $\mathbb{C}_\infty$ as a subset of $\mathbb{P}^1(\mathbb{C}_\infty)$ by

$$\mathbb{C}_\infty \to \mathbb{P}^1(\mathbb{C}_\infty), \quad z \mapsto (1 : -z).$$

Thus we define $\infty = (0 : 1)$.

This is compatible with the Möbius transformation, namely

$$(1 : -\gamma(z)) = \left(1 : -\frac{az + b}{cz + d}\right) = (1 : -z)\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \gamma\circ(1 : -z).$$

(3) For any vertex $v = \gamma \circ v_0$ in the Bruhat–Tits tree $\mathcal{T}$, we define $U(v) = \gamma \circ U(v_0)$.

(4) For any edge $e = (v \to w)$, we define in Definition 11.4 the orientation of the annulus $V(e)$ as follows: we have closed discs $U(e)$ and $U(-e)$ in $\mathbb{P}^1(\mathbb{C}_\infty)$ satisfying

$$V(e) = \mathbb{P}^1(\mathbb{C}_\infty)\backslash(U(e) \sqcup U(-e)).$$

Then the orientation of $V(e)$ is given by an isomorphism

$$w : V(e) \to \{z \in \mathbb{C}_\infty \mid 1 < |z| < q\}$$

such that $w^*(z)$ extends to a rigid analytic function on $U(e)$ having zero at the center of $U(e)$.

In general there are two consistent choices of orientations of $V(e)$ for all e, but this is the unique choice that makes $\mathrm{Res}(F) = c$ holds true.

(5) The residue map is defined as

$$\mathrm{Res}(f)(e)(X^iY^{n-2-i}) = \mathrm{Res}_e((-z)^{n-2-i}f(z)dz).$$

(6) The measure associated with a harmonic cocycle is defined from the formula

$$\int_{U(e)} x^i d\mu_c(x) = (-1)^i c(e)(X^{n-2-i}Y^i)$$
$$\Leftrightarrow \int_{U(e)} (-x)^{n-2-i} d\mu_c(x) = c(e)(X^i Y^{n-2-i}).$$

1.3. **Notation.** Let p be a rational prime, $q > 1$ be a p-power integer and $\mathbb{F}_q$ be the field of q elements. Let t be an indeterminate. Put $A = \mathbb{F}_q[t]$ and $K = \mathbb{F}_q(t)$. We denote by A_+ the set of monic polynomials in A.

Let $\pi_\infty = 1/t$ and $K_\infty = \mathbb{F}_q((1/t)) = \mathbb{F}_q((\pi_\infty))$. Let $v_\infty : K_\infty \to \mathbb{Z} \cup \{+\infty\}$ be the normalized additive valuation on K_∞. It is defined by $v_\infty(0) = +\infty$ and

$$v_\infty(a) = r \quad \text{if } a = \sum_{i \geqslant r} c_i \pi_\infty, \ c_r \neq 0.$$

Let $\mathbb{C}_\infty$ be the π_∞-adic completion of an algebraic closure of K_∞. The unique extension of v_∞ to $\mathbb{C}_\infty$ is also denoted by v_∞. For any $z \in \mathbb{C}_\infty$, put

$$|z| = \begin{cases} q^{-v_\infty(z)} & (z \neq 0), \\ 0 & (z = 0). \end{cases}$$

For any field L equipped with an additive valuation v and any rational number $s \geqslant 0$, we put

$$m_L^{\geqslant s} = \{x \in L \mid v(x) \geqslant s\}.$$

For any affinoid algebra R with its supremum seminorm $|-|_{\sup}$, define

$$R^\circ = \{f \in R \mid |f|_{\sup} \leqslant 1\}, \quad R^\vee = \{f \in R \mid |f|_{\sup} < 1\}, \quad \tilde{R} = R^\circ / R^\vee.$$

2. Bruhat–Tits tree

2.1. **The projective line and the action of GL_2.** Let B be an A-algebra. Let $V(B) = B^2$ be the set of row vectors over B. The group $GL_2(B)$ acts naturally from the right via

$$(x, y)\gamma = (ax + cy, bx + dy), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(B).$$

Define its left action on $V(B)$ by

$$\gamma \circ (x, y) = (x, y)\gamma^{-1} = \frac{1}{ad - bc}(dx - cy, -bx + ay).$$

Let L be a field. The multiplicative group $L^\times$ acts on $V(L)$ by

$$c(x,y) = (cx, cy), \quad (x,y) \in V(L), \quad c \in L^\times.$$

Write $\mathbb{P}^1(L) = (V(L)\backslash\{(0,0)\})/L^\times$ and the class of (x,y) by $(x:y)$. Then the action $\circ$ induces an action of $GL_2(L)$ on $\mathbb{P}^1(L)$ which is denoted also by $\circ$.

We consider L as a subset of $\mathbb{P}^1(L)$ by the map

$$\iota : L \to \mathbb{P}^1(L), \quad z \mapsto (1:-z). \tag{2.1}$$

Then we have $\mathbb{P}^1(L) = L \cup \{\infty\}$ with $\infty = (0:1)$.

Lemma 2.1. *The subset $\iota(L) \subseteq \mathbb{P}^1(L)$ is open. Moreover, if we consider $\iota(L)$ as a subspace of $\mathbb{P}^1(L)$, then the map ι induces a homeomorphism $\iota : L \to \iota(L)$.*

Proof. Let $f : L^2\backslash\{(0,0)\} \to \mathbb{P}^1(L)$ be the natural surjection. Since

$$\iota(L) = \{(x:y) \in \mathbb{P}^1(L) \mid x \neq 0\}, \quad f^{-1}(\iota(L)) = \{(x,y) \in L^2\backslash\{(0,0)\} \mid x \neq 0\},$$

we see that $\iota(L)$ is open in $\mathbb{P}^1(L)$.

Since the map ι factors as

$$L \to L^2\backslash\{(0,0)\} \to \mathbb{P}^1(L), \quad z \mapsto f((1,-z)),$$

the map $\iota : L \to \iota(L)$ is continuous. Its inverse map is given by

$$j : \iota(L) \to L, \quad (x:y) \mapsto -yx^{-1}.$$

To show that j is continuous, it is enough to prove that for any open subset $V \subseteq L$, the subset $j^{-1}(V) \subseteq \mathbb{P}^1(L)$ is open. Since the set

$$f^{-1}(j^{-1}(V)) = \{(x,y) \in L^\times \times L \mid -yx^{-1} \in V\}$$

is open in $L^\times \times L$, it is open in $L^2\backslash\{(0,0)\}$ and thus $j^{-1}(V)$ is open in $\mathbb{P}^1(L)$. □

Via the map ι we identify the action $\circ$ of $GL_2(L)$ on $\mathbb{P}^1(L)$ with the Möbius transformation

$$\gamma(z) = \frac{az+b}{cz+d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(L).$$

We give $\mathbb{P}^1(K_\infty) = (V(K_\infty)\backslash\{(0,0)\})/K_\infty^\times$ the quotient topology induced from the natural topology on $V(K_\infty) = K_\infty^2$. Since the natural map

$$(\mathcal{O}_\infty \times \{1\}) \cup (\{1\} \times \mathcal{O}_\infty) \to \mathbb{P}^1(K_\infty)$$

is surjective, it follows that $\mathbb{P}^1(K_\infty)$ is compact.

2.2. **Definition of the tree.** Put $V_\infty = V(K_\infty)$. An $\mathcal{O}_\infty$-lattice of V_∞ is a finitely generated $\mathcal{O}_\infty$-submodule of V_∞ which generates V_∞ over K_∞. We say two $\mathcal{O}_\infty$-lattices M and M' are equivalent if $M' = cM$ for some $c \in K_\infty^\times$, and the equivalence class of M is denoted by $[M]$. We say two equivalence classes Λ and Λ' are adjacent if there exists representatives $M \in \Lambda$ and $M' \in \Lambda'$ such that $M \supseteq M'$ and the $\mathcal{O}_\infty$-module M/M' is of length one.

Let M and M' be two $\mathcal{O}_\infty$-lattices in V_∞. By the elementary divisor theorem, we can find a basis f_1, f_2 of the $\mathcal{O}_\infty$-module M and $a, b \in \mathbb{Z}$ such that $\pi_\infty^a f_1, \pi_\infty^b f_2$ form a basis of M'. The integers a, b do not depend on the choice of a basis of M, and the integer $|a-b|$ depends only on the equivalence classes Λ, Λ' of M, M'. We write

$$\chi(M, M') = a + b, \quad d(\Lambda, \Lambda') = |a - b|$$

and call the latter the distance of Λ and Λ'. Then Λ, Λ' are adjacent if and only if $d(\Lambda, \Lambda') = 1$.

For any representative M of Λ, there exists a unique representative M' of Λ' satisfying $M \supseteq M'$ and the $\mathcal{O}_\infty$-module M/M' is monogenic. Then $d(\Lambda, \Lambda')$ agrees with the length $l(M/M')$ of the $\mathcal{O}_\infty$-module M/M'.

Define a graph $\mathcal{T}$ as follows.

- The set of vertices $\mathcal{T}_0$ is the set of equivalence classes of lattices in V_∞.
- The set of edges $\mathcal{T}_1$ is the set of $\{\Lambda, \Lambda'\}$ consisting of adjacent equivalence classes Λ, Λ'.

Lemma 2.2 ([Ser], Ch. II, §1.1, Theorem 1)**.** *Let n be a positive integer. If $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$ is a sequence of adjacent vertices without backtracking (that is, Λ_i, Λ_{i+1} are adjacent and $\Lambda_i \neq \Lambda_{i+2}$ for any i), then $d(\Lambda_0, \Lambda_n) = n$.*

Proof. We proceed by induction on n. For $n = 1$ it is trivial. Suppose $n \geqslant 2$. We can find representatives M_i of Λ_i satisfying $M_i \supseteq M_{i+1}$ and $l(M_i/M_{i+1}) = 1$. Then $l(M_0/M_n) = n$. By the definition of the distance, it is enough to show $M_n \nsubseteq \pi_\infty M_0$.

By the induction hypothesis we have $d(\Lambda_0, \Lambda_{n-1}) = n-1$ and $M_{n-1} \nsubseteq \pi_\infty M_0$. By $l(M_{n-2}/M_{n-1}) = 1$, we have $\pi_\infty M_{n-2} \subseteq M_{n-1}$ and $l(M_{n-1}/\pi_\infty M_{n-2}) = 1$. Thus the image of $\pi_\infty M_{n-2}$ in the $\mathbb{F}_q$-vector space $M_{n-1}/\pi_\infty M_{n-1}$ is a one-dimensional subspace.

On the other hand, the image of M_n in $M_{n-1}/\pi_\infty M_{n-1}$ is also one-dimensional. Indeed, if the image is zero, then we have $M_n \subseteq \pi_\infty M_{n-1} \subseteq M_{n-1}$, which contradicts $l(M_{n-1}/M_n) = 1$. If the image has dimension two, then we have $M_{n-1} = M_n + \pi_\infty M_{n-1}$ and Nakayama's lemma implies $M_{n-1} = M_n$, which is also a contradiction.

If these one-dimensional subspaces agree, then we have

$$M_n + \pi_\infty M_{n-1} = \pi_\infty M_{n-2} + \pi_\infty M_{n-1} = \pi_\infty M_{n-2}.$$

Since $l(M_{n-1}/M_n) = 1$, we have $\pi_\infty M_{n-1} \subseteq M_n$ and this shows $M_n = \pi_\infty M_{n-2}$. Thus it gives the backtracking $\Lambda_n = \Lambda_{n-2}$, which is a contradiction. Hence the images of M_n and $\pi_\infty M_{n-2}$ generate the $\mathcal{O}_\infty$-module $M_{n-1}/\pi_\infty M_{n-1}$, namely $M_{n-1} = M_n + \pi_\infty M_{n-2}$. This forces $M_n \nsubseteq \pi_\infty M_0$. □

Using Lemma 2.2, we can show that $\mathcal{T}$ is a connected $(q+1)$-regular tree and the distance $d(\Lambda, \Lambda')$ of equivalence classes Λ, Λ' agrees with the distance of the corresponding vertices in the tree $\mathcal{T}$.

We call $\mathcal{T}$ the Bruhat–Tits tree (for $PGL_2(K_\infty)$). To any edge $\{v, w\} \in \mathcal{T}_1$, we attach two oriented edges $(v \to w)$ and $(w \to v)$. The set of oriented edges is denoted by $\mathcal{T}_1^o$. We refer to an element of $\mathcal{T}_0 \sqcup \mathcal{T}_1^o$ a simplex of $\mathcal{T}$.

For any oriented edge $e = (v \to w)$, we denote its reverse edge $(w \to v)$ by $-e$, its origin v by $o(e)$ and its terminus w by $t(e)$. Then the action $\circ$ of the group $GL_2(K_\infty)$ on V_∞ induces its action on $\mathcal{T}_0$, and also on $\mathcal{T}_1^o$ by $\gamma \circ (v \to w) = (\gamma \circ v \to \gamma \circ w)$. Then the actions on $\mathcal{T}_0$ and $\mathcal{T}_1^o$ are both transitive.

Lemma 2.3. *Let $\Lambda \in \mathcal{T}_0$ be any vertex of $\mathcal{T}$ represented by an $\mathcal{O}_\infty$-lattice M of V_∞. Let $\gamma \in GL_2(K_\infty)$. Then we have*

$$\chi(M, \gamma \circ M) = v_\infty(\det(\gamma)), \quad d(\Lambda, \gamma \circ \Lambda) \equiv v_\infty(\det(\gamma)) \bmod 2.$$

Proof. Choose a basis e_1, e_2 of the $\mathcal{O}_\infty$-module M. Then there exist integers $a, b \in \mathbb{Z}$ such that $\pi_\infty^a e_1, \pi_\infty^b e_2$ form a basis of the $\mathcal{O}_\infty$-module $\gamma \circ M$. Since $v_\infty(\det(\gamma)) = a + b$, we obtain the first equality and

$$d(\Lambda, \gamma \circ \Lambda) = |a - b| \equiv a + b = v_\infty(\det(\gamma)) \bmod 2.$$

□

Definition 2.4. For any $i \in \mathbb{Z}$, put $f_1 = (1, 0)$, $f_2 = (0, 1)$ and

$$v_0 = [\mathcal{O}_\infty f_1 \oplus \mathcal{O}_\infty f_2], \quad v_i = \begin{pmatrix} \pi_\infty^{-i} & 0 \\ 0 & 1 \end{pmatrix} \circ v_0 = [\mathcal{O}_\infty \pi_\infty^i f_1 \oplus \mathcal{O}_\infty f_2].$$

Then v_i and v_{i+1} are adjacent. Put

$$e_i = (v_i \to v_{i+1}).$$

We call v_i and e_i the i-th standard vertex and edge.

We can show

$$\begin{aligned} \mathrm{Stab}_{GL_2(K_\infty)}(v_0) &= GL_2(\mathcal{O}_\infty)K_\infty^\times, \\ \mathrm{Stab}_{GL_2(K_\infty)}(e_0) &= K_0(\pi_\infty)K_\infty^\times, \end{aligned} \tag{2.2}$$

where we put

$$K_0(\pi_\infty) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\infty) \;\middle|\; c \in \pi_\infty \mathcal{O}_\infty \right\}.$$

Thus we obtain bijections [GN, (1.1)]

$$\begin{aligned} GL_2(K_\infty)/GL_2(\mathcal{O}_\infty)K_\infty^\times \to \mathcal{T}_0, &\quad \gamma \mapsto \gamma \circ v_0, \\ GL_2(K_\infty)/K_0(\pi_\infty)K_\infty^\times \to \mathcal{T}_1^o, &\quad \gamma \mapsto \gamma \circ e_0. \end{aligned} \tag{2.3}$$

Example 2.5. There are exactly $q+1$ vertices which are adjacent to v_0. Any such vertex is represented by the lattice which is the inverse image of a line in $\mathbb{F}_q f_1 \oplus \mathbb{F}_q f_2$. The inverse image of the line $\mathbb{F}_q f_2$ represents v_1. The other lines are $\mathbb{F}_q(f_1 + \lambda f_2)$ with some $\lambda \in \mathbb{F}_q$ and thus the rest of the vertices are

$$[\mathcal{O}_\infty(f_1 + \lambda f_2) \oplus \mathcal{O}_\infty \pi_\infty f_2] = \begin{pmatrix} \pi_\infty & -\lambda \\ 0 & 1 \end{pmatrix} \circ v_0 \quad (\lambda \in \mathbb{F}_q).$$

Example 2.6. Put

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{F}_q).$$

Then $J \circ v_0 = v_0$. On the other hand, we have

$$\begin{pmatrix} \pi_\infty & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \pi_\infty & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & \pi_\infty \\ -\pi_\infty & 0 \end{pmatrix} \in GL_2(\mathcal{O}_\infty)K_\infty^\times,$$

which implies $J \circ v_1 = v_{-1}$ and $J \circ e_0 = -e_{-1}$.

Example 2.7. There are exactly $q+1$ edges whose origin is v_0. One of these edges is e_0. By Example 2.5, we have

$$\begin{pmatrix} \pi_\infty & -\lambda \\ 0 & 1 \end{pmatrix} \circ v_0 = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \circ v_{-1}$$

and the other q edges are

$$\begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} \circ (-e_{-1}) = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix} J \circ e_0 = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \circ e_0 \quad (\lambda \in \mathbb{F}_q).$$

2.3. **Limit points.** A half-line is a sequence $\{v_i\}_{i\geqslant 0}$ of adjacent vertices of $\mathcal{T}$ without backtracking. We say two half-lines $\{v_i\}_{i\geqslant 0}, \{v'_i\}_{i\geqslant 0}$ are equivalent if they are equal except finitely many vertices. An end of $\mathcal{T}$ is an equivalence class of half-lines. The set of ends of $\mathcal{T}$ is denoted by $\mathrm{End}(\mathcal{T})$, on which $GL_2(K_\infty)$ acts naturally via the action $\circ$.

Lemma 2.8 ([FvdP1], (V. 1.12))**.** *Let $\{M_s\}_{s\geqslant 0}$ be a family of $\mathcal{O}_\infty$-lattices in V_∞ such that*

$$M_s \supseteq M_{s+1}, \quad M_0/M_s \simeq \mathcal{O}_\infty/\pi_\infty \mathcal{O}_\infty^s \tag{2.4}$$

for any s. Then $\bigcap_{s\geqslant 0} M_s$ is a direct summand of the $\mathcal{O}_\infty$-module M_0.

Proof. By the elementary divisor theorem, we can find a basis f_1^s, f_2^s of M_0 satisfying

$$M_0 = \mathcal{O}_\infty f_1^s \oplus \mathcal{O}_\infty f_2^s, \quad M_s = \mathcal{O}_\infty \pi_\infty^s f_1^s \oplus \mathcal{O}_\infty f_2^s.$$

Moreover, we may assume

$$f_2^{s+1} - f_2^s \in m_\infty^s f_1^s. \tag{2.5}$$

Indeed, since $f_2^{s+1} \in M_{s+1} \subseteq M_s$ we can write

$$f_2^{s+1} = \pi_\infty^s x f_1^s + y f_2^s, \quad x, y \in \mathcal{O}_\infty.$$

If $y \in m_\infty$, then we have $f_2^{s+1} \in \pi_\infty M_0$, which contradicts (2.4). Thus $y \in \mathcal{O}_\infty^\times$ and replacing f_2^{s+1} with $y^{-1} f_2^{s+1}$ shows the claim.

Since $M_s \subseteq V_\infty$ is closed, the sequence $\{f_2^s\}_{s\geqslant 0}$ converges to some element $f_2 \in \bigcap_{s\geqslant 0} M_s$. Moreover (2.5) yields $f_2^{s+1} - f_2^s \in m_\infty^s M_0$ and $f_2^{s+l} - f_2^s \in m_\infty^s M_0$ for any integer $l \geqslant 0$. By taking the limit we obtain

$$f_2 - f_2^s \in \pi_\infty^s M_0 \quad \text{for any } s. \tag{2.6}$$

In particular, Nakayama's lemma implies that for any $s > 0$, the elements f_1^s, f_2 form a basis of M_0 satisfying

$$M_s = \mathcal{O}_\infty \pi_\infty^s f_1^s \oplus \mathcal{O}_\infty f_2. \tag{2.7}$$

We claim $\bigcap_{s\geqslant 0} M_s = \mathcal{O}_\infty f_2$. Indeed, for any $f \in \bigcap_{s\geqslant 0} M_s$ we write

$$f = \pi_\infty^s x_s f_1^s + y_s f_2^s, \quad x_s, y_s \in \mathcal{O}_\infty.$$

Then (2.6) gives $f - y_s f_2 \in \pi_\infty^s M_0$ and thus $y_s - y_{s+1} \in \pi_\infty^s \mathcal{O}_\infty$ for any $s > 0$. This implies that $\{y_s\}_{s\geqslant 0}$ converges to some $y \in \mathcal{O}_\infty$ satisfying $f = y f_2$, which concludes the proof of the lemma. $\square$

Definition 2.9. Let H be a half-line in $\mathcal{T}$. Write $H = \{[M_s]\}_{s\geqslant 0}$ with lattices M_s satisfying (2.4). By Lemma 2.8, the K-subspace

$$W_\infty = K_\infty \otimes_{\mathcal{O}_\infty} \left(\bigcap_{s\geqslant 0} M_s \right) \subseteq V_\infty$$

is one-dimensional and depends only on the end b represented by H. Now we define

$$\lim(b) = \lim(H) \in \mathbb{P}^1(K_\infty)$$

as the element corresponding to the line W_∞. Then the map lim is $GL_2(K_\infty)$-equivariant.

Lemma 2.10 ([FvdP1], (V. 1.12)). *The map* lim *defines a* $GL_2(K_\infty)$-*equivariant bijection*

$$\lim : \mathrm{End}(\mathcal{T}) \to \mathbb{P}^1(K_\infty).$$

Proof. For any element $z \in \mathbb{P}^1(K_\infty)$ let W_∞ be the corresponding line in V_∞. Let M_0 be a lattice in V_∞. Then $N_0 = W_\infty \cap M_0$ is a direct summand of M_0 of rank one. Put $M_s = N_0 + \pi_\infty^s M_0$. Then $d([M_0],[M_s]) = s$ and $\{[M_s]\}_{s \geqslant 0}$ defines a half-line. Let b_z be the end it defines.

Now $\lim(b_z)$ is the element of $\mathbb{P}^1(K_\infty)$ that $K_\infty \otimes_{\mathcal{O}_\infty} N_0$ defines, namely z. Conversely, for any $b \in \mathrm{End}(\mathcal{T})$ and $z = \lim(b)$, take a family of lattices $\{M_s\}_{s \geqslant 0}$ satisfying (2.4) and defining a half-line which represents b. Then $z \in \mathbb{P}^1(K_\infty)$ is defined by $K_\infty \otimes_{\mathcal{O}_\infty} \bigcap_{s \geqslant 0} M_s$ and (2.7) yields $b_z = b$. This concludes the proof. □

Definition 2.11. We say an end $b \in \mathrm{End}(\mathcal{T})$ is rational if $\lim(b) \in \mathbb{P}^1(K)$. The subset of $\mathrm{End}(\mathcal{T})$ consisting of rational ends is denoted by $\mathrm{End}_K(\mathcal{T})$.

Example 2.12. Consider the half-line $H = \{v_i\}_{i \geqslant 0}$, where the standard vertex v_i is represented by the lattice $M_i = \mathcal{O}_\infty \pi_\infty^i f_1 \oplus \mathcal{O}_\infty f_2$. Then $\bigcap_{i \geqslant 0} M_i = \mathcal{O}_\infty f_2$ and thus $\lim(H) = (0:1) = \infty$.

3. Arithmetic subgroups and cusps

3.1. **Arithmetic subgroups.**

Definition 3.1. We say an A-submodule Y of K^2 is an A-lattice of K^2 if it is a finite projective A-module satisfying $K \otimes_A Y = K^2$.

Let Y be any A-lattice of K^2 and let $\mathfrak{n} \subseteq A$ be any nonzero ideal. Put

$$\Gamma(Y) = \{\gamma \in GL_2(K) \mid \gamma \circ Y = Y\}, \quad \Gamma(Y,\mathfrak{n}) = \{\gamma \in \Gamma(Y) \mid \gamma \equiv \mathrm{id} \bmod \mathfrak{n}Y\}.$$

For $Y = A^2$, we have $\Gamma(Y) = GL_2(A)$ and $\Gamma(Y,\mathfrak{n}) = \Gamma(\mathfrak{n})$, where

$$\Gamma(\mathfrak{n}) = \left\{ \gamma \in GL_2(A) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \mathfrak{n} \right\}.$$

Definition 3.2. We say a subgroup Γ of $GL_2(K)$ is an arithmetic subgroup (with respect to Y) if there exist an A-lattice Y of K^2 and a nonzero ideal $\mathfrak{n} \subseteq A$ satisfying

$$\Gamma(Y, \mathfrak{n}) \subseteq \Gamma \subseteq \Gamma(Y).$$

Since $\Gamma(Y, \mathfrak{n})$ is of finite index in $\Gamma(Y)$, we see that the indices $[\Gamma(Y) : \Gamma]$ and $[\Gamma : \Gamma(Y, \mathfrak{n})]$ are both finite. Moreover, we have

$$\det(\Gamma) \subseteq \mathbb{F}_q^\times.$$

An arithmetic subgroup of $GL_2(K)$ with respect to $Y = A^2$ is called a congruence subgroup.

Note that for any arithmetic subgroup Γ and any $\gamma \in GL_2(K)$, the conjugate $\gamma^{-1}\Gamma\gamma$ is also an arithmetic subgroup associated with the A-lattice $\gamma^{-1} \circ Y$. Moreover, since $A = \mathbb{F}_q[t]$ in our setting, any A-lattice Y in K^2 is free as an A-module and thus any arithmetic subgroup is conjugate to a congruence subgroup.

Lemma 3.3. *Let Y, Y' be any A-lattices of K^2 and let $\mathfrak{n}$ be any nonzero ideal of A. Then there exists a nonzero ideal $\mathfrak{n}' \subseteq \mathfrak{n}$ of A satisfying $\Gamma(Y', \mathfrak{n}') \subseteq \Gamma(Y, \mathfrak{n})$.*

Proof. Replacing Y' by a scalar multiple, we may assume $Y \subseteq Y'$. Take any nonzero ideal $\mathfrak{m} \subseteq \mathfrak{n}$ satisfying $\mathfrak{m}Y' \subseteq Y$. Then we have $\mathfrak{m}^2 Y' \subseteq \mathfrak{m}Y \subseteq \mathfrak{n}Y$. For any $\gamma \in \Gamma(Y', \mathfrak{m}^2)$ and $y \in Y \subseteq Y'$, this yields $\gamma \circ y - y \in \mathfrak{m}^2 Y' \subseteq \mathfrak{n}Y$. In particular, this shows that Y is stable under γ and γ^{-1}. Hence we obtain $\gamma \in \Gamma(Y, \mathfrak{n})$. □

Lemma 3.4 ([Böc], Proposition 3.14)**.** *Let Γ be any subgroup of $GL_2(K)$. Then Γ is an arithmetic subgroup if and only if there exists a nonzero ideal $\mathfrak{n}$ of A such that Γ contains $\Gamma(\mathfrak{n})$ as a subgroup of finite index.*

Proof. Suppose that Γ is an arithmetic subgroup. Take Y and $\mathfrak{n}$ as in Definition 3.2. By Lemma 3.3, for some nonzero ideals $\mathfrak{n}', \mathfrak{n}''$ of A we have

$$\Gamma(Y, \mathfrak{n}'') \subseteq \Gamma(\mathfrak{n}') \subseteq \Gamma(Y, \mathfrak{n}) \subseteq \Gamma.$$

Since $[\Gamma(Y, \mathfrak{n}) : \Gamma(Y, \mathfrak{n}'')]$ is finite, the subgroup $\Gamma(\mathfrak{n}')$ is also of finite index in Γ.

Conversely, suppose that $\Gamma(\mathfrak{n})$ is a subgroup of Γ of finite index for some nonzero ideal $\mathfrak{n}$ of A. Define

$$Y = \bigcap_{\gamma \in \Gamma} \gamma \circ A^2.$$

Since $\gamma \circ A^2 = A^2$ for any $\gamma \in \Gamma(\mathfrak{n})$, the A-module Y is the intersection of subsets $\gamma \circ A^2$ for finitely many $\gamma \in \Gamma$. Thus Y is an A-lattice in K^2

satisfying $\Gamma \subseteq \Gamma(Y)$. By Lemma 3.3, we can find a nonzero ideal $\mathfrak{n}'$ of A satisfying $\Gamma(Y, \mathfrak{n}') \subseteq \Gamma(\mathfrak{n}) \subseteq \Gamma$. Thus Γ is an arithmetic subgroup. $\square$

Note that for any $e \in \mathcal{T}_1^o$, we have

$$\mathrm{Stab}_\Gamma(e) = \mathrm{Stab}_\Gamma(o(e)) \cap \mathrm{Stab}_\Gamma(t(e)) = \mathrm{Stab}_\Gamma(-e). \tag{3.1}$$

Moreover, for any $g \in GL_2(K)$ and any vertex (*resp.* edge) s of $\mathcal{T}$, we have

$$\mathrm{Stab}_{g\Gamma g^{-1}}(g \circ s) = g\mathrm{Stab}_\Gamma(s)g^{-1}. \tag{3.2}$$

Lemma 3.5. *Let G be a subgroup of $GL_2(K_\infty)$ satisfying $|\det(g)| = 1$ for any $g \in G$. Let $v \in \mathcal{T}_0$ and let M be an $\mathcal{O}_\infty$-lattice representing v. Then we have* $\mathrm{Stab}_G(v) = \mathrm{Stab}_G(M)$.

Proof. Since we have $\mathrm{Stab}_G(v) \supseteq \mathrm{Stab}_G(M)$, it is enough to show the reverse containment. Take any $g \in \mathrm{Stab}_G(v)$, so that $g \circ M = xM$ for some $x \in K_\infty^\times$. Then Lemma 2.3 yields

$$2v_\infty(x) = \chi(M, xM) = \chi(M, g \circ M) = v_\infty(g) = 0.$$

Hence $x \in \mathcal{O}_\infty^\times$ and $g \circ M = M$. $\square$

Lemma 3.6. *Let Γ be any arithmetic subgroup of $GL_2(K)$. For any simplex s of $\mathcal{T}$, the stabilizer subgroup* $\mathrm{Stab}_\Gamma(s)$ *is finite.*

Proof. By (3.1), we may assume $s = v \in \mathcal{T}_0$. Let M be an $\mathcal{O}_\infty$-lattice in K_∞^2 representing the vertex v. Since $\det(\Gamma) \subseteq \mathbb{F}_q^\times$, Lemma 3.5 implies $\mathrm{Stab}_\Gamma(v) = \mathrm{Stab}_\Gamma(M)$.

We claim that $\mathrm{Stab}_\Gamma(M)$ is a bounded subset of $GL_2(K_\infty)$. Indeed, take any $\gamma \in GL_2(K_\infty)$ satisfying $\gamma \circ M = \mathcal{O}_\infty^2$ and any $m \in \mathbb{Z}$ satisfying $\pi_\infty^m \gamma, \pi_\infty^m \gamma^{-1} \in M_2(\mathcal{O}_\infty)$. Then, for any $g \in \mathrm{Stab}_\Gamma(M)$, we have $g = \gamma^{-1} h \gamma$ with some $h \in GL_2(\mathcal{O}_\infty)$ and thus $\pi_\infty^{2m} g \in M_2(\mathcal{O}_\infty)$. This yields the claim.

Since $A = \mathbb{F}_q[t]$, we see that Γ is conjugate to a subgroup of $GL_2(A)$. By the claim, $\mathrm{Stab}_\Gamma(v)$ is in bijection with a bounded subset of $GL_2(A)$, which is finite. This concludes the proof. $\square$

Definition 3.7 ([Ser], p. 131)**.** An arithmetic subgroup Γ of $GL_2(K)$ is said to be p'-torsion free if any element of Γ of finite order has a p-power order. Note that if Γ is p'-torsion free, then so is its conjugate.

3.2. **Cusps of arithmetic subgroups and the quotient graph.**

Definition 3.8. For any arithmetic subgroup Γ, let

$$\mathrm{Cusps}(\Gamma) := \Gamma \backslash \mathbb{P}^1(K).$$

We refer to any element of $\mathrm{Cusps}(\Gamma)$ a cusp of Γ.

Lemma 3.9. *Let $\gamma \in GL_2(K_\infty)$ be any element satisfying* $\det(\gamma) \in \mathbb{F}_q^\times$. *Then the action of γ on $\mathcal{T}$ is without inversion. Namely, for any $e \in \mathcal{T}_1^o$ we have $\gamma \circ e \neq -e$.*

Proof. Write $e = (v \to w)$. If $\gamma \circ e = -e$, then we have $\gamma \circ v = w$ and $d(v, \gamma \circ v) = 1$. By Lemma 2.3, this contradicts the assumption $\det(\gamma) \in \mathbb{F}_q^\times$. ☐

Let Γ be an arithmetic subgroup of $GL_2(K)$. Then we have $\det(\Gamma) \subseteq \mathbb{F}_q^\times$. By Lemma 3.9, the action of Γ on $\mathcal{T}$ is without inversion and thus we can define the quotient graph $\Gamma \backslash \mathcal{T}$. Indeed, we define the set of vertices of $\Gamma \backslash \mathcal{T}$ as

$$(\Gamma \backslash \mathcal{T})_0 := \Gamma \backslash \mathcal{T}_0$$

and the set of oriented edges as

$$(\Gamma \backslash \mathcal{T})_1^o := \Gamma \backslash \mathcal{T}_1^o.$$

For any $[e] \in \Gamma \backslash \mathcal{T}_1^o$ which is represented by $e \in \mathcal{T}_1^o$, we define

$$o([e]) := [o(e)], \quad t([e]) := [t(e)], \quad -[e] := [-e].$$

Then the assumption of being without inversion implies $-[e] \neq [e]$ for any $e \in \mathcal{T}_1^o$ and thus $\Gamma \backslash \mathcal{T}$ is a graph.

Put $G = GL_2(A)$, which is an arithmetic subgroup of $GL_2(K)$. Define its subgroups

$$G_0 = GL_2(\mathbb{F}_q), \quad G_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{F}_q^\times, \deg(b) \leqslant n \right\} \ (n \geqslant 1).$$

We describe the quotient graph $G \backslash \mathcal{T}$, following [Ser, Ch. II, §1.6].

Lemma 3.10. *(1) For any $n \geqslant 0$ and $m > 0$, the vertices v_n and v_{n+m} are not equivalent modulo G.*
(2) $\mathrm{Stab}_G(v_n) = G_n$.
(3) G_0 acts transitively on the set $\{e \in \mathcal{T}_1^o \mid o(e) = v_0\}$.
(4) For any $n \geqslant 1$, the group G_n fixes e_n and acts transitively on the set $\{e \in \mathcal{T}_1^o \mid o(e) = v_n\} \backslash \{e_n\}$.

Proof. Suppose that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in G$ satisfies $\gamma \circ v_n = v_{n+m}$ with some integer $m \geqslant 0$. Put $M_n = \mathcal{O}_\infty \pi_\infty^n f_1 \oplus \mathcal{O}_\infty f_2$ representing the vertex v_n. Then we have $\gamma \circ M_n = M_{n+m} \pi_\infty^{-h}$ with some $h \in \mathbb{Z}$. By Lemma 2.3, we have

$$0 = v_\infty(\det(\gamma)) = \chi(M_n, \gamma \circ M_n) = m - 2h$$

and $m = 2h$.

The condition $\gamma \circ M_n \subseteq M_{n+2h}\pi_\infty^{-h}$ implies

$$\pi_\infty^n(a,b),\ (c,d) \in \mathcal{O}_\infty \pi_\infty^{n+h}(1,0) \oplus \mathcal{O}_\infty \pi_\infty^{-h}(0,1)$$

and thus we obtain

$$\deg(a) \leqslant -h, \quad \deg(b) \leqslant n+h, \quad \deg(c) \leqslant -n-h, \quad \deg(d) \leqslant h.$$

For $m > 0$, we have $h > 0$ and $a = c = 0$, which is a contradiction. This shows (1).

For $m = 0$, we have $h = 0$ and this implies $a, d \in \mathbb{F}_q$. Moreover, if $n = 0$ then $c, d \in \mathbb{F}_q$. For $n \geqslant 1$, we have $c = 0$ and $\deg(b) \leqslant n$. Thus (2) follows.

Since the set of $\mathcal{O}_\infty$-submodules of M_0 of index q is naturally identified with $\mathbb{P}^1(\mathbb{F}_q)$, the group G_0 acts transitively on it and (3) follows.

For (4), let $n \geqslant 1$. Since $G_n \subseteq G_{n+1}$, the assertion (2) implies that the group G_n fixes $e_n = (v_n \to v_{n+1})$. On the other hand, the action of G_n on $M_n/\pi_\infty M_n$ factors through the homomorphism

$$G_n \to GL_2(\mathbb{F}_q), \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \begin{pmatrix} a & b_n \\ 0 & d \end{pmatrix}$$

where b_n is the coefficient in degree n of b. The image of this homomorphism is the subgroup $B(\mathbb{F}_q)$ of upper triangular matrices, and the natural right action on $\mathbb{P}^1(\mathbb{F}_q)$ of $B(\mathbb{F}_q)$ fixes $(0:1)$ and is transitive on $\mathbb{P}^1(\mathbb{F}_q)\backslash\{(0:1)\}$. Thus we obtain (4). $\square$

Lemma 3.10 also yields

$$\begin{aligned} &\mathrm{Stab}_G(e_0) = G_0 \cap G_1 = B(\mathbb{F}_q) := \begin{pmatrix} \mathbb{F}_q^\times & \mathbb{F}_q \\ 0 & \mathbb{F}_q^\times \end{pmatrix}, \\ &\mathrm{Stab}_G(e_n) = G_n \cap G_{n+1} = G_n \quad (n \geqslant 1). \end{aligned} \tag{3.3}$$

Lemma 3.11. *Let T be the path consisting of vertices $\{v_n\}_{n\geqslant 0}$ and edges $\{\pm e_n\}_{n\geqslant 0}$. Then the quotient graph $GL_2(A)\backslash\mathcal{T}$ is represented by T.*

Proof. Let $\pi : \mathcal{T} \to \mathcal{T}' = GL_2(A)\backslash\mathcal{T}$ be the natural projection. By Lemma 3.10 (1), the map π defines an isomorphism of T onto a subgraph T' of $\mathcal{T}'$. Since $\mathcal{T}$ and $\mathcal{T}'$ are connected, it is enough to show that any edge $\bar{e} \in \mathcal{T}'$ with $o(\bar{e}) \in T'$ is an edge of T'.

Take any $e \in \mathcal{T}_1^o$ satisfying $\pi(e) = \bar{e}$. By assumption we have $o(e) = \gamma \circ v$ with some $v \in T$ and $\gamma \in G$. Replacing e by $\gamma^{-1} \circ e$, we may assume $o(e) = v_n$ with some integer $n \geqslant 0$. If $n = 0$, then Lemma 3.10 (3) implies that e is equivalent to e_0 modulo G_0 and thus $\bar{e} \in T'$. If $n \geqslant 1$, then Lemma 3.10 (4) implies that e is equivalent to e_n or $-e_{n-1}$ and thus $\bar{e} \in T'$. This concludes the proof. $\square$

Let Γ be a congruence subgroup of $G = GL_2(A)$. Consider the natural projection
$$\pi : \Gamma\backslash\mathcal{T} \to G\backslash\mathcal{T},$$
which is a morphism of graphs. For any integer $n \geqslant 0$, let
$$X_n(\Gamma\backslash\mathcal{T}) = \{v \in (\Gamma\backslash\mathcal{T})_0 \mid \pi(v) = Gv_n\},$$
$$Y_n(\Gamma\backslash\mathcal{T}) = \{e \in (\Gamma\backslash\mathcal{T})_1^o \mid \pi(e) = Ge_n\}.$$
An element of these sets is called a vertex or an edge of type n [GN, §1.4]. By Lemma 3.10 and (3.3), we have bijections
$$\begin{aligned} \Gamma\backslash G/G_n &\to X_n(\Gamma\backslash\mathcal{T}), \quad \Gamma g G_n \mapsto \Gamma g \circ v_n, \\ \Gamma\backslash G/(G_n \cap G_{n+1}) &\to Y_n(\Gamma\backslash\mathcal{T}), \quad \Gamma g(G_n \cap G_{n+1}) \mapsto \Gamma g \circ e_n, \end{aligned} \tag{3.4}$$
and thus they are finite sets. Since $G_n \cap G_{n+1} = G_n$ for any $n \geqslant 1$, (3.4) yields a bijection
$$o_n : Y_n(\Gamma\backslash\mathcal{T}) \to X_n(\Gamma\backslash\mathcal{T}), \quad \Gamma g \circ e_n \to \Gamma g \circ v_n. \tag{3.5}$$

Lemma 3.12. *Let Γ be a congruence subgroup of $G = GL_2(A)$ containing $\Gamma(\mathfrak{n})$ with some $\mathfrak{n} \in A\backslash\mathbb{F}_q$. Let $d = \deg(\mathfrak{n}) \geqslant 1$. Then the natural map*
$$t_n : Y_n(\Gamma\backslash\mathcal{T}) \to X_{n+1}(\Gamma\backslash\mathcal{T}), \quad \Gamma g \circ e_n \to \Gamma g \circ v_{n+1} \tag{3.6}$$
is also a bijection for any $n \geqslant d$. In particular, the subgraph of $\Gamma\backslash\mathcal{T}$ consisting of the vertices and the edges of type $\geqslant d$ is the union of $|X_d(\Gamma\backslash\mathcal{T})|$ injective infinite paths.

Proof. Since $\Gamma\backslash G = (\Gamma(\mathfrak{n})\backslash\Gamma)\backslash(\Gamma(\mathfrak{n})\backslash G)$, the right action of G_n on $\Gamma\backslash G$ factors through the natural homomorphism
$$p_n : G_n \to \Gamma(\mathfrak{n})\backslash G \simeq \mathbb{F}_q^\times SL_2(A/(\mathfrak{n})) \subseteq GL_2(A/(\mathfrak{n})).$$
Since $p_n(G_n) = p_{n+1}(G_{n+1})$ for any $n \geqslant d$, the map t_n is a bijection for any $n \geqslant d$. Combining it with (3.5), we see that the vertices and edges of type $\geqslant d$ form $|X_d(\Gamma\backslash\mathcal{T})|$ injective infinite paths. This concludes the proof. □

Lemma 3.13. *Let Γ be an arithmetic subgroup of $GL_2(K)$. Then the quotient graph $\Gamma\backslash\mathcal{T}$ is the union of a finite graph and finitely many injective infinite paths.*

Proof. Replacing Γ with its conjugate by an element of $GL_2(K)$, we may assume that Γ is a congruence subgroup. Then the lemma follows from Lemma 3.12. □

Definition 3.14. Let Γ be a congruence subgroup of $GL_2(A)$ containing $\Gamma(\mathfrak{n})$, where $\mathfrak{n} \in A$ and $\deg(\mathfrak{n}) = d > 0$. We denote by $\mathrm{Cusps}(\Gamma\backslash\mathcal{T})$ the finite set of injective infinite paths given in Lemma 3.12, so that we have a natural identification

$$\varinjlim_{n\geqslant d} X_n(\Gamma\backslash\mathcal{T}) \to \mathrm{Cusps}(\Gamma\backslash\mathcal{T}), \quad \Gamma g \circ v_n \mapsto \{\Gamma g \circ v_n\}_{n\geqslant d},$$

where the direct limit is taken with respect to the bijection

$$t_n \circ o_n^{-1} : X_n(\Gamma\backslash\mathcal{T}) \to X_{n+1}(\Gamma\backslash\mathcal{T}).$$

Lemma 3.15. *Let Γ be a congruence subgroup of $GL_2(A)$ containing $\Gamma(\mathfrak{n})$ with some $\mathfrak{n} \in A\backslash\mathbb{F}_q$ of degree $d > 0$. Then there exists a natural bijection*

$$\mathrm{Cusps}(\Gamma\backslash\mathcal{T}) \to \Gamma\backslash\mathrm{End}_K(\mathcal{T}), \quad \{\Gamma g \circ v_n\}_{n\geqslant d} \mapsto \Gamma[\{g \circ v_n\}]_{n\geqslant 0}.$$

In particular, composing the map lim *of Lemma 2.10 we obtain a natural bijection*

$$\mathrm{Cusps}(\Gamma\backslash\mathcal{T}) \to \Gamma\backslash\mathrm{End}_K(\mathcal{T}) \simeq \mathrm{Cusps}(\Gamma).$$

Proof. Let

$$G_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, d \in \mathbb{F}_q^\times,\ b \in A \right\}.$$

Since the group $GL_2(A)$ acts transitively on $\mathbb{P}^1(K)$, Lemma 2.10 and Example 2.12 yield a bijection

$$GL_2(A)/G_\infty \to \mathrm{End}_K(\mathcal{T}), \quad gG_\infty \mapsto [\{g \circ v_n\}_{n\geqslant 0}].$$

Since the proof of Lemma 3.12 shows that the natural map

$$\Gamma\backslash GL_2(A)/G_n \to \Gamma\backslash GL_2(A)/G_\infty$$

is bijective for any $n \geqslant d$, taking the direct limit yields the lemma. ∎

3.3. **Euler–Poincaré characteristic.** Let Γ be an arithmetic subgroup of $GL_2(K)$. For any $v \in \mathcal{T}_0$ and $e \in \mathcal{T}_1^o$, by Lemma 3.6 we have positive integers

$$|\mathrm{Stab}_\Gamma(v)|, \quad |\mathrm{Stab}_\Gamma(e)|.$$

By (3.1) and (3.2), these numbers satisfy

$$|\mathrm{Stab}_\Gamma(v)| = |\mathrm{Stab}_\Gamma(\gamma \circ v)|, \quad |\mathrm{Stab}_\Gamma(e)| = |\mathrm{Stab}_\Gamma(\pm\gamma \circ e)|$$

for any $\gamma \in \Gamma$. Thus we may consider the positive series

$$\chi_0(\Gamma) = \sum_{v\in\Gamma\backslash\mathcal{T}_0} \frac{1}{|\mathrm{Stab}_\Gamma(v)|}, \quad \chi_1(\Gamma) = \sum_{e\in\Gamma\backslash\mathcal{T}_1^o/\{\pm 1\}} \frac{1}{|\mathrm{Stab}_\Gamma(e)|}.$$

For $G = GL_2(A)$, Lemma 3.11, Lemma 3.10 (2) and (3.3) yield

$$\begin{aligned}\chi_0(G) &= \frac{1}{(q^2-1)(q-q)} + \sum_{n\geqslant 1} \frac{1}{(q-1)^2 q^{n+1}},\\ \chi_1(G) &= \frac{1}{q(q-1)^2} + \sum_{n\geqslant 1} \frac{1}{(q-1)^2 q^{n+1}},\end{aligned} \tag{3.7}$$

which are both convergent.

Lemma 3.16. *The series $\chi_0(\Gamma)$ and $\chi_1(\Gamma)$ are convergent.*

Proof. By (3.2), replacing Γ with its conjugate we may assume that Γ is a congruence subgroup. Put $G = GL_2(A)$. Then we have natural surjections

$$\Gamma\backslash\mathcal{T}_0 \to G\backslash\mathcal{T}_0, \quad \Gamma\backslash\mathcal{T}_1^o/\{\pm 1\} \to G\backslash\mathcal{T}_1^o/\{\pm 1\}$$

whose fibers have at most $[G : \Gamma]$ elements. Moreover, for any simplex s of $\mathcal{T}$, we have $\mathrm{Stab}_\Gamma(s) \subseteq \mathrm{Stab}_G(s)$ and

$$\begin{aligned}[\mathrm{Stab}_G(s) : \mathrm{Stab}_\Gamma(s)] &= [\mathrm{Stab}_G(s) : \Gamma \cap \mathrm{Stab}_G(s)]\\ &= [\Gamma\mathrm{Stab}_G(s) : \mathrm{Stab}_G(s)] \leqslant [G : \Gamma],\end{aligned}$$

which yields

$$\frac{1}{|\mathrm{Stab}_\Gamma(s)|} = \frac{[\mathrm{Stab}_G(s) : \mathrm{Stab}_\Gamma(s)]}{|\mathrm{Stab}_G(s)|} \leqslant \frac{[G : \Gamma]}{|\mathrm{Stab}_G(s)|}.$$

Hence we obtain

$$\chi_i(\Gamma) \leqslant [G : \Gamma]^2 \chi_i(G)$$

and (3.7) implies the convergence for Γ. □

Definition 3.17. We define

$$\chi(\Gamma) = \sum_{v \in \Gamma\backslash\mathcal{T}_0} \frac{1}{|\mathrm{Stab}_\Gamma(v)|} - \sum_{e \in \Gamma\backslash\mathcal{T}_1^o/\{\pm 1\}} \frac{1}{|\mathrm{Stab}_\Gamma(e)|}$$

and call it the Euler–Poincaré characteristic of Γ. By Lemma 3.16, this series is absolutely convergent and

$$\chi(\Gamma) = \chi_0(\Gamma) - \chi_1(\Gamma).$$

By (3.7), we have

$$\chi(GL_2(A)) = -\frac{1}{(q-1)^2(q+1)}. \tag{3.8}$$

Lemma 3.18. *Let Γ' be any arithmetic subgroup of $GL_2(K)$ which is contained in Γ, so that Γ' is of finite index in Γ. Then we have*

$$\chi_i(\Gamma') = [\Gamma : \Gamma']\chi_i(\Gamma) \ (i = 0, 1), \quad \chi(\Gamma') = [\Gamma : \Gamma']\chi(\Gamma).$$

Proof. It is enough to show the equality for χ_i. Replacing Γ by its conjugate, we may assume $\Gamma \subseteq GL_2(A)$. By Lemma 3.4, for some nonzero ideal $\mathfrak{n} \subseteq A$ we have $\Gamma'' := \Gamma(\mathfrak{n}) \subseteq \Gamma'$. Then Γ'' are normal both in Γ and Γ'. If the lemma holds for $\Gamma'' \subseteq \Gamma$ and $\Gamma'' \subseteq \Gamma'$, then we have

$$[\Gamma : \Gamma'']\chi_i(\Gamma) = \chi_i(\Gamma'') = [\Gamma' : \Gamma'']\chi_i(\Gamma'),$$

from which we obtain $\chi_i(\Gamma') = [\Gamma : \Gamma']\chi_i(\Gamma)$. Hence we may assume $\Gamma' \lhd \Gamma$.

Put $X_0 = \mathcal{T}_0$ and $X_1 = \mathcal{T}_1^o/\{\pm 1\}$. Since Γ acts on $\mathcal{T}$ without inversion, for any $e \in \mathcal{T}_1^o$ and its image $[e]$ in X_1, we have $\mathrm{Stab}_\Gamma(e) = \mathrm{Stab}_\Gamma([e])$.

Let Λ_i be a complete set of representatives of $\Gamma\backslash X_i$. For any $\gamma, \delta \in \Gamma$ and $s \in X_i$, since $\Gamma' \lhd \Gamma$ we have

$$\Gamma'\gamma s = \Gamma'\delta s \quad \Leftrightarrow \quad \Gamma'\gamma s \cap \Gamma'\delta s \neq \emptyset \quad \Leftrightarrow \quad \delta \in \gamma\Gamma'\mathrm{Stab}_\Gamma(s).$$

This yields a decomposition

$$X_i = \coprod_{s\in\Lambda_i} \Gamma s = \coprod_{s\in\Lambda_i} \coprod_{\gamma\in\Gamma/\Gamma'\mathrm{Stab}_\Gamma(s)} \Gamma'\gamma s,$$

where the subgroup $\Gamma'\mathrm{Stab}_\Gamma(s)$ is also of finite index in Γ. Thus we have

$$\begin{aligned}\chi_i(\Gamma') &= \sum_{s\in\Lambda_i} \sum_{\gamma\in\Gamma/\Gamma'\mathrm{Stab}_\Gamma(s)} |\mathrm{Stab}_{\Gamma'}(\gamma s)|^{-1} \\ &= \sum_{s\in\Lambda_i} |\mathrm{Stab}_\Gamma(s)|^{-1} \sum_{\gamma\in\Gamma/\Gamma'\mathrm{Stab}_\Gamma(s)} \frac{|\mathrm{Stab}_\Gamma(s)|}{|\mathrm{Stab}_{\Gamma'}(\gamma s)|}.\end{aligned}$$

Since (3.2) yields $|\mathrm{Stab}_{\Gamma'}(\gamma s)| = |\mathrm{Stab}_{\Gamma'}(s)|$, this equals

$$\begin{aligned}&\sum_{s\in\Lambda_i} |\mathrm{Stab}_\Gamma(s)|^{-1} \sum_{\gamma\in\Gamma/\Gamma'\mathrm{Stab}_\Gamma(s)} \frac{|\mathrm{Stab}_\Gamma(s)|}{|\mathrm{Stab}_{\Gamma'}(s)|} \\ &\quad= \sum_{s\in\Lambda_i} |\mathrm{Stab}_\Gamma(s)|^{-1}[\Gamma : \Gamma'\mathrm{Stab}_\Gamma(s)][\mathrm{Stab}_\Gamma(s) : \mathrm{Stab}_{\Gamma'}(s)] \\ &\quad= \sum_{s\in\Lambda_i} |\mathrm{Stab}_\Gamma(s)|^{-1}[\Gamma : \Gamma'\mathrm{Stab}_\Gamma(s)][\mathrm{Stab}_\Gamma(s) : \Gamma' \cap \mathrm{Stab}_\Gamma(s)] \\ &\quad= \sum_{s\in\Lambda_i} |\mathrm{Stab}_\Gamma(s)|^{-1}[\Gamma : \Gamma'\mathrm{Stab}_\Gamma(s)][\Gamma'\mathrm{Stab}_\Gamma(s) : \Gamma'] \\ &\quad= [\Gamma : \Gamma'] \sum_{s\in\Lambda_i} |\mathrm{Stab}_\Gamma(s)|^{-1} = [\Gamma : \Gamma']\chi_i(\Gamma).\end{aligned}$$

This concludes the proof. □

4. Discs and the Bruhat–Tits tree

4.1. **Distinguished closed discs.** Let L be a field equipped with a complete non-Archimedean valuation $|-|: L \to \mathbb{R}_{\geqslant 0}$. We denote by $\mathcal{O}_L$ the ring of integers of L and by m_L the maximal ideal of $\mathcal{O}_L$.

Definition 4.1. For any $a \in L$ and $\rho \in |L^\times|$, consider the following subsets of $\mathbb{P}^1(L)$.

$$D_L(a,\rho) = \{x \in L \mid |x-a| \leqslant \rho\}, \quad D'_L(a,\rho) = \{x \in L \mid |x-a| \geqslant \rho\} \cup \{\infty\}.$$

We refer to them closed discs in $\mathbb{P}^1(L)$. Moreover, we put

$$D^\circ_L(a,\rho) = \{x \in L \mid |x-a| < \rho\}, \quad D'^\circ_L(a,\rho) = \{x \in L \mid |x-a| > \rho\} \cup \{\infty\}.$$

We call them open discs in $\mathbb{P}^1(L)$. We also put

$$C_L(a,\rho) = \{x \in L \mid |x-a| = \rho\}.$$

We refer to a as a center of there discs and the circle. When $L = K_\infty$, we often drop the subscript L as $D(a,\rho)$ or $D'(a,\rho)$.

We also write

$$D_L(\infty,\rho) := D'_L(0,\rho^{-1}), \quad D^\circ_L(\infty,\rho) := D'^\circ_L(0,\rho^{-1}).$$

Lemma 4.2. *Let $a, a' \in L$ and $\rho, \rho' \in |L^\times|$ satisfying $\rho \geqslant \rho'$.*

(1) $D_L(a,\rho) \cap D_L(a',\rho') \neq \varnothing$ if and only if $|a-a'| \leqslant \rho$. In this case, we have $D_L(a',\rho') \subseteq D_L(a,\rho)$.
(2) $D^\circ_L(a,\rho) \cap D^\circ_L(a',\rho') \neq \varnothing$ if and only if $|a-a'| < \rho$. In this case, we have $D^\circ_L(a',\rho') \subseteq D^\circ_L(a,\rho)$.

Proof. If $|a-a'| \leqslant \rho$, then $a' \in D_L(a,\rho) \cap D_L(a',\rho')$. Moreover, for any $z \in D_L(a',\rho')$, we have

$$|z-a| = |z-a'+(a'-a)| \leqslant \max\{\rho',\rho\} = \rho$$

and thus $D_L(a',\rho') \subseteq D_L(a,\rho)$. Conversely, if $|a-a'| > \rho$, then for any $z \in D_L(a',\rho')$ we have

$$|z-a| = |z-a'+(a'-a)| = |a'-a| > \rho$$

and thus $D_L(a,\rho) \cap D_L(a',\rho') \neq \varnothing$, which shows (1). The assertion (2) follows similarly. $\square$

Lemma 4.3. *Let $a, a' \in L$ and $\rho, \rho' \in |L^\times|$. If $D_L(a,\rho) \subseteq D_L(a',\rho')$ or $D'_L(a,\rho) \supseteq D'_L(a',\rho')$, Then $\rho \leqslant \rho'$. In particular,*

$$D_L(a,\rho) = D_L(a',\rho') \text{ or } D'_L(a,\rho) = D'_L(a',\rho') \quad \Rightarrow \quad \rho = \rho'.$$

Proof. Suppose $D_L(a,\rho) \subseteq D_L(a',\rho')$ and $\rho' < \rho$. Take any $x \in L$ satisfying $|x-a| = \rho$. Then $|a-a'| \leqslant \rho' < \rho$ and $x \in D_L(a,\rho) \subseteq D_L(a',\rho')$. Hence we have

$$\rho' \geqslant |x-a'| = |x-a+(a-a')| = |x-a| = \rho,$$

which is a contradiction.

Suppose $D'_L(a,\rho) \supseteq D'_L(a',\rho')$ and $\rho' < \rho$. Taking the complement we have $D^\circ_L(a,\rho) \subseteq D^\circ_L(a',\rho')$. Take any $x \in L$ satisfying $|x-a| = \rho'$. Then $|a-a'| < \rho'$ and $x \in D^\circ_L(a,\rho) \subseteq D^\circ_L(a',\rho')$. Hence we have

$$\rho' > |x-a'| = |x-a+(a-a')| = |x-a| = \rho',$$

which is a contradiction. $\square$

Definition 4.4. Let $L \in \{K_\infty, \mathbb{C}_\infty\}$. For any $a \in K_\infty$ and $\rho \in |K_\infty^\times| = q^{\mathbb{Z}}$, we refer to the closed discs in $\mathbb{P}^1(L)$.

$$D_L(a,\rho) = \{x \in L \mid |x-a| \leqslant \rho\}, \quad D'_L(a,\rho) = \{x \in L \mid |x-a| \geqslant \rho\} \cup \{\infty\}$$

as distinguished closed discs in $\mathbb{P}^1(L)$.

The set of distinguished closed discs in $\mathbb{P}^1(L)$ is denoted by $\mathrm{DCD}(L)$.

Lemma 4.5. *For any $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$, we have*

$$\begin{aligned} D(a,\rho) \subseteq D(a',\rho') &\quad\Leftrightarrow\quad D_{\mathbb{C}_\infty}(a,\rho) \subseteq D_{\mathbb{C}_\infty}(a',\rho'), \\ D'(a,\rho) \subseteq D'(a',\rho') &\quad\Leftrightarrow\quad D'_{\mathbb{C}_\infty}(a,\rho) \subseteq D'_{\mathbb{C}_\infty}(a',\rho'). \end{aligned}$$

Proof. Suppose $D(a,\rho) \subseteq D(a',\rho')$. Then Lemma 4.3 yields $\rho \leqslant \rho'$. Since $|a-a'| \leqslant \rho'$, by Lemma 4.2 (1) we obtain $D_{\mathbb{C}_\infty}(a,\rho) \subseteq D_{\mathbb{C}_\infty}(a',\rho')$.

Suppose $D'(a,\rho) \subseteq D'(a',\rho')$ so that $D^\circ(a,\rho) \supseteq D^\circ(a',\rho')$. Then Lemma 4.3 yields $\rho \geqslant \rho'$. Since $|a-a'| < \rho$, by Lemma 4.2 (2) we obtain $D^\circ_{\mathbb{C}_\infty}(a,\rho) \supseteq D^\circ_{\mathbb{C}_\infty}(a',\rho')$ and $D'_{\mathbb{C}_\infty}(a,\rho) \subseteq D'_{\mathbb{C}_\infty}(a',\rho')$.

The other implications follow from

$$D(a,\rho) = D_{\mathbb{C}_\infty}(a,\rho) \cap \mathbb{P}^1(K_\infty), \quad D'(a,\rho) = D'_{\mathbb{C}_\infty}(a,\rho) \cap \mathbb{P}^1(K_\infty).$$

$\square$

By Lemma 4.5, we have a well-defined bijection

$$\mathrm{DCD}(K_\infty) \to \mathrm{DCD}(\mathbb{C}_\infty), \quad D(a,\rho) \mapsto D_{\mathbb{C}_\infty}(a,\rho),\ D'(a,\rho) \mapsto D'_{\mathbb{C}_\infty}(a,\rho).$$

For any $D \in \mathrm{DCD}(K_\infty)$, we call its image in $\mathrm{DCD}(\mathbb{C}_\infty)$ the extension of D over $\mathbb{C}_\infty$.

Lemma 4.6. *Let $D_1, D_2 \in \mathrm{DCD}(K_\infty)$. Let $D_{1,\mathbb{C}_\infty}$ and $D_{2,\mathbb{C}_\infty}$ be their extensions over $\mathbb{C}_\infty$. Then*

$$D_1 \cap D_2 = \varnothing \quad\Leftrightarrow\quad D_{1,\mathbb{C}_\infty} \cap D_{2,\mathbb{C}_\infty} = \varnothing.$$

Proof. Suppose $D_1 \cap D_2 = \varnothing$. Since $\infty \in D'(a,\rho)$, replacing D_1 and D_2 if necessary we may assume

$$D_1 = D(a_1,\rho_1), \quad D_2 \in \{D(a_2,\rho_2),\ D'(a_2,\rho_2)\}$$

with some $a_1, a_2 \in K_\infty$ and $\rho_1, \rho_2 \in q^{\mathbb{Z}}$.

Suppose $D_2 = D(a_2,\rho_2)$. Then we have $|a_1 - a_2| > \max\{\rho_1,\rho_2\}$. If $z \in D_{1,\mathbb{C}_\infty} \cap D_{2,\mathbb{C}_\infty}$, then

$$|a_1 - a_2| = |z - a_1 - (z - a_2)| \leqslant \max\{\rho_1,\rho_2\},$$

which is a contradiction.

Suppose $D_2 = D'(a_2,\rho_2)$. Then $a_1 \notin D'(a_2,\rho_2)$ and $|a_1 - a_2| < \rho_2$. If $z \in D_{1,\mathbb{C}_\infty} \cap D_{2,\mathbb{C}_\infty}$, then

$$\rho_1 \geqslant |z - a_1| = |z - a_2 + (a_2 - a_1)| = |z - a_2| \geqslant \rho_2.$$

This forces $\rho_2 \leqslant \rho_1$. Take any $\varpi_{\rho_2} \in K_\infty$ satisfying $|\varpi_{\rho_2}| = \rho_2$. Then $z = a_1 + \varpi_{\rho_2} \in K_\infty$ satisfies

$$|z - a_1| = \rho_2 \leqslant \rho_1, \quad |z - a_2| = |a_1 - a_2 + \varpi_{\rho_2}| = \rho_2$$

and thus $z \in D_1 \cap D_2$, which is a contradiction. Hence we obtain $D_{1,\mathbb{C}_\infty} \cap D_{2,\mathbb{C}_\infty} = \varnothing$.

Since $D_1 \cap D_2 \subseteq D_{1,\mathbb{C}_\infty} \cap D_{2,\mathbb{C}_\infty}$, the converse is clear. $\square$

Consider the action of $GL_2(L)$ on $\mathbb{P}^1(L)$ via the Möbius transformation as before.

Lemma 4.7. *Let $\rho \in |L^\times|$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(L)$. Let $z \in L$. Then we have*

$$\gamma(C_L(z,\rho)) = \begin{cases} C_L\left(\gamma(z), \rho\frac{|ad-bc|}{|cz+d|^2}\right) & (|cz+d| > \rho|c|), \\ C_L\left(\frac{a}{c}, \frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right) & (|cz+d| < \rho|c|). \end{cases}$$

Proof. Write $\gamma^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

First suppose $|cz+d| > \rho|c|$. Then we have $cz+d \neq 0$ and $\gamma(z) \neq \infty$. For any $x \in L$ with $|x - z| = \rho$, it follows that

$$|(cx+d) - (cz+d)| = |c(x-z)| = \rho|c|, \quad |cx+d| = |cz+d| > 0$$

and thus

$$\left|\frac{ax+b}{cx+d} - \frac{az+b}{cz+d}\right| = \frac{|ad-bc||x-z|}{|cz+d||cx+d|} = \frac{|ad-bc|}{|cz+d|^2}|x-z|.$$

This yields

$$\gamma(C_L(z,\rho)) \subseteq C_L\left(\gamma(z), \rho\frac{|ad-bc|}{|cz+d|^2}\right).$$

Since γ^{-1} satisfies

$$c'\gamma(z)+d'=\frac{1}{cz+d},\quad |c'\gamma(z)+d'|=\frac{1}{|cz+d|}>\rho\frac{|ad-bc|}{|cz+d|^2}|c'|,$$

applying the containment above to γ^{-1} shows

$$\gamma^{-1}\left(C_L\left(\gamma(z),\rho\frac{|ad-bc|}{|cz+d|^2}\right)\right)\subseteq C_L(z,\rho),$$

which yields the lemma for this case.

Next we suppose $|cz+d|<\rho|c|$. Since $ad-bc\neq 0$, we have $c\neq 0$ and $\gamma(-\frac{d}{c})=\infty$ with

$$\left|-\frac{d}{c}-z\right|=\frac{|cz+d|}{|c|}<\rho. \tag{4.1}$$

For any $x\in L$ with $|x-z|=\rho$, it follows that

$$|(cx+d)-(cz+d)|=|c(x-z)|=\rho|c|,\quad |cx+d|=\rho|c|$$

and

$$\left|\frac{ax+b}{cx+d}-\frac{a}{c}\right|=\frac{|ad-bc|}{|c||cx+d|}=\frac{1}{\rho}\frac{|ad-bc|}{|c|^2},$$

which gives

$$\gamma(C_L(z,\rho))\subseteq C_L\left(\frac{a}{c},\frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right).$$

Since we have $c'(\frac{a}{c})+d'=0$, applying this to γ^{-1} we have

$$\gamma^{-1}\left(C_L\left(\frac{a}{c},\frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right)\right)\subseteq C_L\left(-\frac{d}{c},\rho\right).$$

By (4.1), we have

$$C_L\left(-\frac{d}{c},\rho\right)=C_L(z,\rho)$$

and the reverse containment also follows. □

Lemma 4.8. *Let $\rho\in|L^\times|$ and $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in GL_2(L)$. Let $z\in L$. Then we have*

$$\gamma(D_L(z,\rho))=\begin{cases}D_L\left(\gamma(z),\rho\frac{|ad-bc|}{|cz+d|^2}\right) & (|cz+d|>\rho|c|),\\ D'_L\left(\frac{a}{c},\frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right) & (|cz+d|\leqslant\rho|c|).\end{cases}$$

In particular, for any $z \in L$ and any $\varpi_\rho \in L^\times$ satisfying $|\varpi_\rho| = \rho$, we have

$$D_L(z,\rho) = \gamma(D_L(0,1)), \quad \gamma = \begin{pmatrix} \varpi_\rho & z \\ 0 & 1 \end{pmatrix},$$
$$D'_L(z,\rho) = \gamma'(D_L(0,1)), \quad \gamma' = \begin{pmatrix} z & \varpi_\rho \\ 1 & 0 \end{pmatrix}.$$

Proof. First suppose $|cz+d| > \rho|c|$. Then $\gamma(z) \neq \infty$. By Lemma 4.7, we have

$$\begin{aligned}
\gamma(D_L(z,\rho)) &= \{\gamma(z)\} \cup \bigcup_{\sigma\in(0,\rho]\cap|L^\times|} \gamma(C_L(z,\sigma)) \\
&= \{\gamma(z)\} \cup \bigcup_{\sigma\in(0,\rho]\cap|L^\times|} C_L\left(\gamma(z), \sigma\frac{|ad-bc|}{|cz+d|^2}\right) \\
&= D_L\left(\gamma(z), \rho\frac{|ad-bc|}{|cz+d|^2}\right).
\end{aligned}$$

Next we suppose $|cz+d| \leqslant \rho|c|$. Since $ad-bc \neq 0$, we have $c \neq 0$ and $\gamma(\infty) = \frac{a}{c}$. Moreover, for any $\rho < \sigma$, we have $|cz+d| < \sigma|c|$. By Lemma 4.7, we have

$$\begin{aligned}
\gamma(D_L(z,\rho)) &= \gamma\left(\mathbb{P}^1(L)\backslash D'^{\circ}_L(z,\rho)\right) \\
&= \gamma\left(\mathbb{P}^1(L)\backslash\left(\{\infty\} \cup \bigcup_{\sigma\in(\rho,+\infty)\cap|L^\times|} C_L(z,\sigma)\right)\right) \\
&= \mathbb{P}^1(L)\backslash\left(\left\{\frac{a}{c}\right\} \cup \bigcup_{\sigma\in(\rho,+\infty)\cap|L^\times|} \gamma(C_L(z,\sigma))\right) \\
&= \mathbb{P}^1(L)\backslash\left(\left\{\frac{a}{c}\right\} \cup \bigcup_{\sigma\in(\rho,+\infty)\cap|L^\times|} C_L\left(\frac{a}{c}, \frac{1}{\sigma}\frac{|ad-bc|}{|c|^2}\right)\right) \\
&= \mathbb{P}^1(L)\backslash D^{\circ}_L\left(\frac{a}{c}, \frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right) \\
&= D'_L\left(\frac{a}{c}, \frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right).
\end{aligned}$$

$\square$

Lemma 4.9. *Let* $\rho \in |L^\times|$ *and* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(L)$. *Let* $z \in L$. *Then we have*

$$\gamma(D'_L(z,\rho)) = \begin{cases} D'_L\left(\gamma(z), \rho\frac{|ad-bc|}{|cz+d|^2}\right) & (|cz+d| \geqslant \rho|c|), \\ D_L\left(\frac{a}{c}, \frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right) & (|cz+d| < \rho|c|). \end{cases}$$

Proof. First we suppose $|cz+d| \geqslant \rho|c|$. Then $cz+d \neq 0$ and $\gamma(z) \neq \infty$. Note that if $\rho > \sigma$ and $|cz+d| \geqslant \rho|c|$, then we have $|cz+d| > \sigma|c|$ even when $c = 0$. By Lemma 4.7, we have

$$\begin{aligned} \gamma(D'_L(z,\rho)) &= \gamma\left(\mathbb{P}^1(L)\backslash D^\circ_L(z,\rho)\right) \\ &= \gamma\left(\mathbb{P}^1(L)\backslash\left(\{z\} \cup \bigcup_{\sigma\in(0,\rho)\cap|L^\times|} C_L(z,\sigma)\right)\right) \\ &= \mathbb{P}^1(L)\backslash\left(\{\gamma(z)\} \cup \bigcup_{\sigma\in(0,\rho)\cap|L^\times|} \gamma(C_L(z,\sigma))\right) \\ &= \mathbb{P}^1(L)\backslash\left(\{\gamma(z)\} \cup \bigcup_{\sigma\in(0,\rho)\cap|L^\times|} C_L\left(\gamma(z), \sigma\frac{|ad-bc|}{|cz+d|^2}\right)\right) \\ &= \mathbb{P}^1(L)\backslash D^\circ_L\left(\gamma(z), \rho\frac{|ad-bc|}{|cz+d|^2}\right) \\ &= D'_L\left(\gamma(z), \rho\frac{|ad-bc|}{|cz+d|^2}\right). \end{aligned}$$

Next suppose $|cz+d| < \rho|c|$. Then we have $c \neq 0$ and $\gamma(\infty) = \frac{a}{c}$. By Lemma 4.7, we have

$$\begin{aligned} \gamma(D'_L(z,\rho)) &= \gamma\left(\{\infty\} \cup \bigcup_{\sigma\in[\rho,+\infty)\cap|L^\times|} C_L(z,\sigma)\right) \\ &= \left\{\frac{a}{c}\right\} \cup \bigcup_{\sigma\in[\rho,+\infty)\cap|L^\times|} \gamma(C_L(z,\sigma)) \\ &= \left\{\frac{a}{c}\right\} \cup \bigcup_{\sigma\in[\rho,+\infty)\cap|L^\times|} C_L\left(\frac{a}{c}, \frac{1}{\sigma}\frac{|ad-bc|}{|c|^2}\right) \\ &= D_L\left(\frac{a}{c}, \frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right). \end{aligned}$$

□

Lemma 4.10. *The group $GL_2(K_\infty)$ acts transitively on* $\mathrm{DCD}(L)$*. Moreover, the extension map*

$$\mathrm{DCD}(K_\infty) \to \mathrm{DCD}(\mathbb{C}_\infty), \quad D \mapsto D_{\mathbb{C}_\infty}$$

is $GL_2(K_\infty)$-equivariant.

Proof. Lemma 4.8 and Lemma 4.9 imply that the image of a distinguished closed disc by any element of $GL_2(K_\infty)$ is again a distinguished closed disc, and also the action is transitive. Since the formulas in these lemmas are independent of the choice of $L \in \{K_\infty, \mathbb{C}_\infty\}$, the $GL_2(K_\infty)$-equivariance of the extension map follows. □

Lemma 4.11. *The stabilizer in $GL_2(K_\infty)$ of $D_L(0,1)$ is $K_0(\pi_\infty)K_\infty^\times$. The same holds for $D'_L(0,q)$.*

Proof. By Lemma 4.10, we may assume $L = K_\infty$. Since $\mathbb{P}^1(K_\infty) = D(0,1) \sqcup D'(0,q)$, it is enough to show the assertion on $D(0,1)$.

Take any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_\infty)$. By Lemma 4.8 and Lemma 4.3, we have $\gamma(D(0,1)) = D(0,1)$ if and only if

$$|d| > |c|, \quad \frac{|ad - bc|}{|d|^2} = 1 \quad \text{and} \quad \left|\frac{b}{d}\right| \leqslant 1,$$

namely if $\gamma \in K_0(\pi_\infty)K_\infty^\times$. □

Corollary 4.12. *We have a $GL_2(K_\infty)$-equivariant bijection*

$$GL_2(K_\infty)/K_0(\pi_\infty)K_\infty^\times \to \mathrm{DCD}(K_\infty), \quad \gamma \mapsto \gamma(D'(0,q)).$$

Lemma 4.13. *The set* $\mathrm{DCD}(K_\infty)$ *forms an open base of the topology of* $\mathbb{P}^1(K_\infty)$*.*

Proof. Since the action on $\mathbb{P}^1(K_\infty)$ of $GL_2(K_\infty)$ is continuous, by Lemma 4.10 it is enough to consider an open neighborhood U of $0 = (1:0)$. Let $p : K_\infty^2 \backslash \{(0,0)\} \to \mathbb{P}^1(K_\infty)$ be the natural projection. Since $p^{-1}(U)$ is an open neighborhood of $(1,0)$ in $K_\infty^2 \backslash \{(0,0)\}$, we can find positive integers m, n satisfying

$$(1,0) \in (1 + \pi_\infty^m \mathcal{O}_\infty) \times \pi_\infty^n \mathcal{O}_\infty \subseteq p^{-1}(U).$$

Then we have $0 \in \{x \in K_\infty \mid |x| \leqslant q^{-n}\} \subseteq U$. □

Lemma 4.14. *Let $\nu, \nu' \in \mathbb{Z}$ and let $a \in K_\infty$.*

(1) *Suppose $\nu > -\nu'$. If $|a| \geqslant q^\nu$, then $D(a, q^{-\nu'}) \subseteq D(\infty, q^{-\nu})$.*
(2) *Suppose $\nu' \geqslant 1$. If $|a| \leqslant q^{\nu'-1}$, then $D(a, q^{-\nu'}) \subseteq D(0, q^{\nu'-1})$.*

In particular, if $\nu' \geqslant 1 + |\nu|$, then there exists a finite subset $\Lambda \subseteq K_\infty$ satisfying

$$\{x \in K_\infty \mid q^\nu \leqslant |x| \leqslant q^{\nu'-1}\} = \coprod_{a \in \Lambda} D(a, q^{-\nu'}).$$

Proof. Suppose $\nu > -\nu'$ and $|a| \geqslant q^\nu$. If $x \in K_\infty$ satisfies $|x| < q^\nu$, then $|x - a| = |a| \geqslant q^\nu > q^{-\nu'}$ and $x \notin D(a, q^{-\nu'})$. This shows (1).

Suppose $\nu' \geqslant 1$ and $|a| \leqslant q^{\nu'-1}$. For any $x \in D(a, q^{-\nu'})$, we have

$$|x| = |x - a + a| \leqslant \max\{q^{-\nu'}, q^{\nu'-1}\} \leqslant q^{\nu'-1},$$

which shows (2). If $\nu' \geqslant 1+|\nu|$, then $\nu' \geqslant 1$ and $\nu' > |\nu| = \max\{\nu, -\nu\}$, which yields $\nu > -\nu'$ and $\nu' > \nu$. Thus the last assertion follows from (1) and (2). □

Lemma 4.15. *Let U be a compact open subset of $\mathbb{P}^1(K_\infty)$. Then there exist $a_1, \ldots, a_r \in K_\infty$ and $\rho_1, \ldots, \rho_r, \rho \in q^{\mathbb{Z}}$ satisfying*

$$U = \begin{cases} \coprod_{i=1}^r D(a_i, \rho_i) & (\infty \notin U), \\ \coprod_{i=1}^r D(a_i, \rho_i) \sqcup D(\infty, \rho) & (\infty \in U). \end{cases}$$

Proof. By Lemma 4.13, we see that U is a finite union of elements of $\mathrm{DCD}(K_\infty)$. If $\infty \notin U$, then we have $U = \bigcup_{i=1}^s D(a_i, \rho_i)$ with some $a_i \in K_\infty$ and $\rho_i \in q^{\mathbb{Z}}$. Then Lemma 4.2 implies that by taking a subcovering we can make this union disjoint.

Suppose $\infty \in U$. We can find a sufficiently large $\rho \in q^{\mathbb{Z}}$ satisfying $D(\infty, \rho) \subseteq U$. Then $U \backslash D(\infty, \rho) = U \cap D(0, q^{-1}\rho^{-1})$ is also compact, and the lemma follows from the former part of the proof. □

Lemma 4.16. *Let $a_1, \ldots, a_r$ and $a'_1, \ldots, a'_s$ be elements of $\mathbb{P}^1(K_\infty)$ and let $\rho_1, \ldots, \rho_r$ and $\rho'_1, \ldots, \rho'_s$ be elements of $q^{\mathbb{Z}}$. Suppose that we have the equality*

$$U := \coprod_{i=1}^r D(a_i, \rho_i) = \coprod_{j=1}^s D(a'_j, \rho'_j).$$

Then there exists a finite covering

$$U = \coprod_{\lambda \in \Lambda} D(a_\lambda, \rho)$$

consisting of distinguished closed discs in $\mathbb{P}^1(K_\infty)$ such that it is a refinement of the two coverings above.

Proof. Take any $\rho \in q^{\mathbb{Z}}$ satisfying $\rho < \min\{\rho_i, \rho_i^{-1}, \rho'_j, (\rho'_j)^{-1}, 1\}$ for any i, j. Then we have $\rho \leqslant \frac{1}{q\rho}$ and

$$b \in D(a_i, \rho_i) \quad \Rightarrow \quad D(b, \rho) \subseteq D(a_i, \rho_i) \tag{4.2}$$

even when b or a_i is ∞. If $a_i \neq \infty$, then we can choose a finite subset $\Lambda_i \subseteq K_\infty$ satisfying

$$D(a_i, \rho_i) = \coprod_{a \in \Lambda_i} D(a, \rho).$$

If $a_i = \infty$, then the choice of ρ implies that for any $b \in D(\infty, \rho_i) \backslash D(\infty, \rho)$ we have

$$z \in D(b, \rho) \quad \Rightarrow \quad z \in D(\infty, \rho_i) \backslash D(\infty, \rho).$$

Thus we can find a finite $\Lambda_i' \subseteq K_\infty$ satisfying

$$D(\infty, \rho_i) = D(\infty, \rho) \sqcup \coprod_{a \in \Lambda_i'} D(a, \rho).$$

In this case, we put $\Lambda_i = \Lambda_i' \cup \{\infty\}$.

Hence the covering

$$U = \coprod_{i=1}^{r} \coprod_{a \in \Lambda_i} D(a, \rho)$$

is a refinement of the first covering of U. For any $i = 1, \ldots, r$ and $a \in \Lambda_i$, we have $a \in D(a_j', \rho_j')$ with some $j = 1, \ldots, s$. Then (4.2) yields $D(a, \rho) \subseteq D(a_j', \rho_j')$. Thus it is also a refinement of the second covering. □

4.2. **Distinguished closed discs and edges in the tree.**

Definition 4.17. For any $e \in \mathcal{T}_1^o$, let $H(e)$ be the set of half-lines H in $\mathcal{T}$ such that H starts from $o(e)$ and passes $t(e)$. This means $H = \{w_n\}_{n \geqslant 0}$ with $e = (w_0 \to w_1)$. Define

$$U(e) = \{\lim(H) \mid H \in H(e)\}.$$

Since the map lim is $GL_2(K_\infty)$-equivariant, we have

$$\gamma(U(e)) = U(\gamma \circ e) \quad \text{for any } \gamma \in GL_2(K_\infty). \tag{4.3}$$

From the definition, for any $v \in \mathcal{T}_0$ we obtain

$$\mathbb{P}^1(K_\infty) = U(e) \sqcup U(-e), \quad \mathbb{P}^1(K_\infty) = \coprod_{o(e)=v} U(e). \tag{4.4}$$

Moreover, for any $e \in \mathcal{T}_1^o$ we have

$$U(e) = \coprod_{o(e')=t(e),\ e' \neq -e} U(e'). \tag{4.5}$$

Lemma 4.18 ([FvdP1], (V.1.13))**.**

$$U(e_0)=\{(x_1:x_2)\in\mathbb{P}^1(K_\infty)\mid |x_1|<|x_2|\}.$$

Via the identification (2.1), this implies

$$U(e_0)=\{x\in K_\infty\mid |x|>1\}\cup\{\infty\}=D'(0,q).$$

Proof. Let $f_1=(1,0)$, $f_2=(0,1)$ be the standard basis of V_∞ and let

$$M_0=\mathcal{O}_\infty f_1\oplus\mathcal{O}_\infty f_2,\quad M_1=\mathcal{O}_\infty\pi_\infty f_1\oplus\mathcal{O}_\infty f_2$$

be representatives of the vertices v_0 and v_1, respectively, By the proof of Lemma 2.8, an element of $\mathbb{P}^1(K_\infty)$ lies in $U(e_0)$ if and only if it corresponds to the line $K_\infty\otimes_{\mathcal{O}_\infty}N$ with some direct summand N of the $\mathcal{O}_\infty$-module M_0 which is contained also in M_1. It is the same as saying $N=\mathcal{O}_\infty(x_1f_1+x_2f_2)$ with some $x_1\in m_\infty$ and $x_2\in\mathcal{O}_\infty^\times$. This yields the lemma. □

Note that Lemma 4.18 and (4.3) yield $U(e)\in\mathrm{DCD}(K_\infty)$ for any $e\in\mathcal{T}_1^o$.

Definition 4.19. We denote by $\mathcal{U}(e)\in\mathrm{DCD}(\mathbb{C}_\infty)$ the extension of $U(e)$ over $\mathbb{C}_\infty$. Namely,

$$\mathcal{U}(e)=\begin{cases}\{z\in\mathbb{C}_\infty\mid |z-a|\leqslant\rho\} & (U(e)=D(a,\rho)),\\ \{z\in\mathbb{C}_\infty\mid |z-a|\geqslant\rho\}\cup\{\infty\} & (U(e)=D'(a,\rho)).\end{cases}$$

For any edges $e,e'\in\mathcal{T}_1^o$, the definition of $U(e)$ implies

$$U(e)\supseteq U(e'),\quad \mathcal{U}(e)\supseteq\mathcal{U}(e')\quad\text{if } o(e')=t(e). \tag{4.6}$$

Moreover, by (4.3) and Lemma 4.10 we have

$$\gamma(\mathcal{U}(e))=\mathcal{U}(\gamma\circ e)\quad\text{for any }\gamma\in GL_2(K_\infty). \tag{4.7}$$

Lemma 4.20. *The map $e\mapsto U(e)$ defines a $GL_2(K_\infty)$-equivariant bijection $\mathcal{T}_1^o\to\mathrm{DCD}(K_\infty)$.*

Proof. By Lemma 4.18, this map sends e_0 to $D'(0,q)$. Then the lemma follows from (2.3) and Corollary 4.12. □

Example 4.21. By Example 2.7 and Lemma 4.18, the closed discs corresponding to edges whose origin is v_0 are

$$U(e_0)=\{x\in K_\infty\mid |x|\geqslant q\}\cup\{\infty\},$$

$$\left(\begin{pmatrix}1&-\lambda\\0&1\end{pmatrix}J\right)(U(e_0))=\{x\in K_\infty\mid |x+\lambda|\leqslant q^{-1}\}\quad(\lambda\in\mathbb{F}_q).$$

Example 4.22. For the edge e_i, we have

$$U(e_i) = \begin{pmatrix} \pi_\infty^{-i} & 0 \\ 0 & 1 \end{pmatrix} (U(e_0)) = \{x \in K_\infty \mid |x| \geqslant q^{i+1}\} \cup \{\infty\}.$$

This yields

$$U(-e_i) = \{x \in K_\infty \mid |x| \leqslant q^i\}.$$

Lemma 4.23. *Let $H = \{w_i\}_{i \geqslant 0}$ be a half-line in $\mathcal{T}$ and let $e'_i = (w_i \to w_{i+1})$. Then we have*

$$\{\lim(H)\} = \bigcap_{i \geqslant 0} \mathcal{U}(e'_i).$$

Proof. By Lemma 2.10, translating by the action of $GL_2(K_\infty)$ we may assume $H = \{v_i\}_{i \in \mathbb{Z}_{\geqslant 0}}$. Then we have $\lim(H) = \infty$ by Example 2.12. Now Example 4.22 yields

$$\mathcal{U}(e_i) = \{z \in \mathbb{C}_\infty \mid |z| \geqslant q^{i+1}\} \cup \{\infty\},$$

from which the lemma follows. $\square$

Definition 4.24. For any $e \in \mathcal{T}_1^o$ and $a \in K_\infty$, define

$$\rho(e) = \begin{cases} \rho & (U(e) = D(a, \rho)), \\ \rho^{-1} & (U(e) = D'(a, \rho)). \end{cases}$$

Note that when $U(e) = D(\infty, \rho) = D'(0, \rho^{-1})$ we have $\rho(e) = \rho$.

By Lemma 4.3, we see that $\rho(e)$ depends only on e.

Lemma 4.25. *Suppose that $e, e' \in \mathcal{T}_1^o$ satisfy $U(e) \supseteq U(e')$. Let $H = \{w_n\}_{n \geqslant 0} \in H(e)$ and $H' = \{w'_n\}_{n \geqslant 0} \in H(e')$ be half-lines satisfying $\lim(H) = \lim(H')$. Then we have $e' = (w_m \to w_{m+1})$ with some $m \geqslant 0$.*

Proof. The assumption $\lim(H) = \lim(H')$ shows that H and H' agree except finitely many vertices. On the other hand, let $P'' = \{w''_n\}_{n=0}^m$ be the unique path without backtracking which satisfies $w''_0 = w_0$ and $w''_m = w'_0$. Since $\mathcal{T}$ has no circuit, we have either $H' = \{w_n\}_{n \geqslant m}$, or $m > 0$ and $H = \{w'_n\}_{n \geqslant m}$. In the former case, we have $e' = (w_m \to w_{m+1})$. In the latter case, any half-line which starts from w'_0 and does not pass w'_1 defines an element of $U(e') \backslash U(e)$, which is a contradiction. $\square$

Lemma 4.26. *Suppose that $e, e'_1, \ldots, e'_r \in \mathcal{T}_1^o$ satisfy*

$$U(e) = \coprod_{i=1}^r U(e'_i).$$

Then for any half-line $H = \{w_n\}_{n \geqslant 0} \in H(e)$*, there exists unique* $i = i(H) \in \{1, \dots, r\}$ *such that* $e'_i = (w_n \to w_{n+1})$ *with some integer* $n \geqslant 0$*. Moreover, the map*

$$H(e) \to \{1, \dots, r\}, \quad H \mapsto i(H)$$

is surjective.

Proof. Take a unique i satisfying $\lim(H) \in U(e'_i)$. By Lemma 4.25, we see that H passes through e'_i.

Suppose that $i, j \in \{1, \dots, r\}$ satisfy the condition of the lemma. Then H passes through both of e'_i and e'_j. Thus we obtain $\lim(H) \in U(e'_i) \cap U(e'_j)$, which is a contradiction.

Take any $i \in \{1, \dots, r\}$ and any $x \in U(e'_i)$. Let $H \in H(e)$ be the half-line representing $x \in U(e)$. By Lemma 4.25, the half-line H passes through e'_i and $i(H) = i$. This concludes the proof of the lemma. $\square$

4.3. **Explicit description of** $U(e)$**.** For any integer $n \geqslant 0$, let

$$\mathcal{T}_0(n) = \{v \in \mathcal{T}_0 \mid d(v_0, v) = n\},$$
$$\mathcal{T}_1^o(n) = \{e \in \mathcal{T}_1^o \mid o(e) \in \mathcal{T}_0(n),\ t(e) \in \mathcal{T}_0(n+1)\}.$$

Then we have

$$\mathbb{P}^1(K_\infty) = \coprod_{e \in \mathcal{T}_1^o(n)} U(e) \tag{4.8}$$

and by Example 4.22 the only disc appearing in this decomposition and containing ∞ is $U(e_n)$.

Definition 4.27. For any local $\mathcal{O}_\infty$-algebra R and a free R-module M of rank two, we denote by $\mathbb{P}^1(M)$ the set of direct summand of rank one of the R-module M.

For the $\mathcal{O}_\infty$-lattice $M_0 = \mathcal{O}_\infty f_1 \oplus \mathcal{O}_\infty f_2$ in V_∞, write $\mathbb{P}^1(\mathcal{O}_\infty) = \mathbb{P}^1(M_0)$.

Consider the natural reduction map

$$p_n : M_0 \to M_0/\pi_\infty^n M_0.$$

Note that for any integers $a, b \in [0, n]$, if $|a - b| = n$ then $(a, b) = (n, 0)$ or $(0, n)$. Now the definition of the distance shows that the map

$$\mathbb{P}^1(M_0/\pi_\infty^n M_0) \to \mathcal{T}_0(n), \quad \bar{N} \mapsto [p_n^{-1}(\bar{N})] \tag{4.9}$$

is a $GL_2(\mathcal{O}_\infty)$-equivariant bijection, and this induces a bijection

$$\mathbb{P}^1(M_0/\pi_\infty^{n+1} M_0) \to \mathcal{T}_1^o(n) \tag{4.10}$$

by sending $\bar{N}$ to the unique edge $e \in \mathcal{T}_1^o(n)$ satisfying $t(e) = [p_{n+1}^{-1}(\bar{N})]$.

Definition 4.28. Let L be K_∞ or $\mathbb{C}_\infty$. For any rational number $s \geqslant 0$, put $M_{0,L} = M_0 \otimes_{\mathcal{O}_\infty} \mathcal{O}_L$ and

$$r_{s,L} : \mathbb{P}^1(L) = \mathbb{P}^1(\mathcal{O}_L) \to \mathbb{P}^1(M_{0,L}/m_L^{\geqslant s} M_{0,L}).$$

We also write $r_s = r_{s,K_\infty}$.

Definition 4.29. A subset Λ of $\mathbb{P}^1(K_\infty)$ is called a complete set of representatives modulo π_∞^n if the restriction to Λ of the natural map

$$r_n : \mathbb{P}^1(K_\infty) = \mathbb{P}^1(\mathcal{O}_\infty) \to \mathbb{P}^1(M_0/\pi_\infty^n M_0),$$

is a bijection.

Definition 4.30. Let $n \geqslant 0$ be an integer and let Λ be a complete set of representatives modulo π_∞^{n+1}. We denote the composite of the inverse maps of r_{n+1} and (4.10) by

$$c_\Lambda : \mathcal{T}_1^o(n) \to \Lambda,$$

which is a bijection.

Lemma 4.31. *Let $n \geqslant 0$ be any integer and let Λ be any complete set of representatives modulo π_∞^{n+1}. For any $e \in \mathcal{T}_1^o(n)$, we have*

$$c_\Lambda(e) \in U(e).$$

Proof. Write $x := c_\Lambda(e) = (x_1 : x_2)$ with $\max\{|x_1|, |x_2|\} = 1$. Let $f_1 = (1,0)$ and $f_2 = (0,1)$ be the standard basis of V_∞ and let $N = \mathcal{O}_\infty(x_1 f_1 + x_2 f_2)$, which is a direct summand of the $\mathcal{O}_\infty$-module M_0.

By the definition of the map (4.10), if we write $e = (v \to w)$ then we have $w = [N + \pi_\infty^{n+1} M_0]$. For any $j \geqslant 0$, put $w_j = [N + \pi_\infty^j M_0]$. Then $H = \{w_j\}_{j \geqslant 0}$ is the unique half-line in $\mathcal{T}$ starting from v_0 which satisfies $\lim(H) = x$. Since e is the unique element of $\mathcal{T}_1^o(n)$ satisfying $t(e) = w$, it follows that H passes through e. Since $\lim(H) = (x_1 : x_2) = x$, we have $x \in U(e)$. □

Definition 4.32. For any $e \in \mathcal{T}_1^o(n)$, we denote by $P(v_0, e) = \{w_i\}_{i=0}^{n+1}$ be the unique path from v_0 passing through e. This means that w_i and w_{i+1} are adjacent vertices for any $i \leqslant n$, $w_0 = v_0$ and $e = (w_n \to w_{n+1})$. We denote by $i(e)$ the maximal integer $i \leqslant n+1$ satisfying $w_i = v_i$.

Lemma 4.33. *Let $n \geqslant 0$ be any integer and let $e \in \mathcal{T}_1^o(n)$. Then we have*

$$\rho(e) = \begin{cases} q^{2i(e)-n-1} & (e \neq e_n), \\ q^{-n-1} & (e = e_n), \end{cases} \qquad i(e) = \begin{cases} \max\{0, \log_q |c_\Lambda(e)|\} & (e \neq e_n), \\ n+1 & (e = e_n). \end{cases}$$

In particular, for any complete set of representatives Λ *modulo* π_∞^{n+1} *we have*

$$U(e)=\begin{cases} D(c_\Lambda(e),q^{2i(e)-n-1}) & (e\neq e_n),\\ D(\infty,q^{-n-1}) & (e=e_n)\end{cases}$$

and $|c_\Lambda(e)|\leqslant q^n$ *if* $e\neq e_n$.

Proof. Since the case of $e=e_n$ follows from Example 4.22, we may assume $e\neq e_n$. Then we have $i(e)\leqslant n$. By Lemma 4.31 the last assertion follows from the assertions on $\rho(e)$ and $i(e)$. It is enough to show these assertions. For this we proceed by induction on n.

Suppose $n=0$. Since $e\neq e_0$, we have $i(e)=0$. Example 4.21 shows $U(e)=D(\lambda,q^{-1})$ for some $\lambda\in\mathbb{F}_q$ and $\rho(e)=q^{-1}$. Lemma 4.31 yields $|c_\Lambda(e)|=1$ and the lemma follows for this case.

Assume that $n\geqslant 1$ and the lemma holds for any element of $\mathcal{T}_1^o(n-1)$. Write $P(v_0,e)=\{w_i\}_{i=0}^{n+1}$ as above.

First suppose $w_n\neq v_n$. Then $i(e)\leqslant n-1$ and the edge $e'=(w_{n-1}\to w_n)$ lies in $\mathcal{T}_1^o(n-1)$ and is not equal to e_{n-1}. Hence $\infty\notin U(e')$ and $i(e')=i(e)$. By the induction hypothesis we have $U(e')=D(a',q^{2i(e)-n})$ with $a'=c_\Lambda(e')$ satisfying $i(e')=\max\{0,\log_q|a'|\}$. Then (4.5) yields

$$U(e')=D(a',q^{2i(e)-n})=\coprod_{e''\in\mathcal{T}_1^o(n),\ o(e'')=w_n}U(e'').$$

Namely, $D(a',q^{2i(e)-n})$ is the disjoint union of q distinguished closed discs $\{D(a_l,\rho_l)\}_{l=1}^q$ with some $a_l\in D(a',q^{2i(e)-n})$ and $\rho_l\in q^{\mathbb{Z}}$ satisfying $\rho_l<q^{2i(e)-n}$. This forces $\rho_l=q^{2i(e)-n-1}$ for any l.

By applying Lemma 4.31 to $e''=e$, we obtain $U(e)=D(a,q^{2i(e)-n-1})$ with $a=c_\Lambda(e)$. Since $U(e)\subseteq U(e')$, we have $|a-a'|\leqslant q^{2i(e)-n}$.

If $|a'|\leqslant 1$ then $i(e')=i(e)=0$ and $a\in D(a',q^{-n})$, which yields $|a|\leqslant 1$ and $\max\{0,\log_q|a|\}=0=i(e)$. If $|a'|>1$, then $|a'|=q^{i(e')}=q^{i(e)}$. Since $i(e)\leqslant n-1$, we have $2i(e)-n<i(e)$ and $|a|=q^{i(e)}$.

Next suppose $w_n=v_n$, which implies $e_{n-1}=(w_{n-1}\to w_n)$. Since $e\neq e_n$, we have $i(e)=n$ and (4.5) shows

$$\{x\in K_\infty\mid |x|=q^n\}=U(e_{n-1})\backslash U(e_n)=\coprod_{e''\in\mathcal{T}_1^o(n),\ o(e'')=w_n,\ e''\neq e_n}U(e'').$$

Thus it is the disjoint union of $q-1$ distinguished closed discs $\{D(a_l,\rho_l)\}_{l=1}^{q-1}$ with some $a_l\in K_\infty$ and $\rho_l\in q^{\mathbb{Z}}$ satisfying $\rho_l<q^n=|a_l|$. This forces $\rho_l=q^{n-1}$ for any l. Since $c_\Lambda(e)\in U(e)$ by Lemma 4.31, we also obtain $|c_\Lambda(e)|=q^n=q^{i(e)}$. This concludes the proof of the lemma. $\square$

We give an example Λ_n of the complete set of representatives modulo π_∞^{n+1}. For any integer $i \in [0,n]$ we define $\Lambda_{i,n} \subseteq K_\infty$ by

$$\Lambda_{i,n} = \begin{cases} \mathbb{F}_q + \mathbb{F}_q\pi_\infty + \cdots + \mathbb{F}_q\pi_\infty^n & (i = 0), \\ \mathbb{F}_q^\times \pi_\infty^{-i} + \mathbb{F}_q\pi_\infty^{-i+1} + \cdots + \mathbb{F}_q\pi_\infty^{n-2i} & (i > 0) \end{cases}$$

and put

$$\Lambda_n = \{\infty\} \sqcup \coprod_{i=0}^{n} \Lambda_{i,n} \subseteq \mathbb{P}^1(K_\infty).$$

Lemma 4.34. *For any integer $n \geqslant 0$, the subset Λ_n is a complete set of representatives modulo π_∞^{n+1}.*

Proof. Put $\mathcal{O}_{\infty,n+1} = \mathcal{O}_\infty/\pi_\infty^{n+1}\mathcal{O}_\infty$. Since we have

$$|\Lambda_n| = |\mathbb{P}^1(M_0/\pi_\infty^{n+1}M_0)| = q^{n+1} + q^n,$$

it is enough to show that the restriction to Λ_n of the map

$$r_n : \mathbb{P}^1(K_\infty) = \mathbb{P}^1(\mathcal{O}_\infty) \to \mathbb{P}^1(M_0/\pi_\infty^n M_0),$$

is surjective.

Let $f_1 = (1,0)$ and $f_2 = (0,1)$. Take any $\bar{N} \in \mathbb{P}^1(M_0/\pi_\infty^{n+1}M_0)$ and $x_1, x_2 \in \mathcal{O}_\infty$ such that the image of $x_1f_1 + x_2f_2$ in $M_0/\pi_\infty^{n+1}M_0$ generates $\bar{N}$.

If $|x_1| = 1$, then we may assume $x_1 = 1$ and $x_2 \in \Lambda_{0,n}$, which implies that $x_2 \in \Lambda_{0,n}$ satisfies $r_{n+1}((1 : x_2)) = \bar{N}$. If $|x_1| \leqslant q^{-n-1}$, then we may assume $(x_1, x_2) = (0,1)$ and $r_{n+1}(\infty) = \bar{N}$.

On the other hand, if $|x_1| \in [q^{-n}, q^{-1}]$, we may assume $x_1 = \pi_\infty^i$ with some integer $i \in [1,n]$ and $x_2 \in \mathcal{O}_\infty^\times$. Note that for any $a, b \in \mathcal{O}_\infty^\times$, we have $\mathcal{O}_{\infty,n+1}^\times(\pi_\infty^i f_1 + af_2) = \mathcal{O}_{\infty,n+1}^\times(\pi_\infty^i f_1 + bf_2)$ if and only if $a \in b(1 + \pi_\infty^{n+1-i}\mathcal{O}_\infty)$, which means $a \equiv b \bmod \pi_\infty^{n+1-i}$. This shows that we may assume $x_2 \in \pi_\infty^i\Lambda_{i,n}$ and thus $r_{n+1}((1 : \pi_\infty^{-i}x_2)) = \bar{N}$. □

4.4. **Description of $U(e)$ via projectivized closed discs.** Let L be K_∞ or $\mathbb{C}_\infty$.

Definition 4.35. For any $\alpha \in \mathbb{P}^1(L)$, its unimodular coordinate is $\alpha = (\alpha_1 : \alpha_2)$ with $\max\{|\alpha_1|, |\alpha_2|\} = 1$.

Lemma 4.36. *For any $z \in \mathbb{P}^1(\mathbb{C}_\infty)$, the map*

$$\mathbb{P}^1(L) \to \mathbb{R}_{\geqslant 0}, \quad \alpha \mapsto |z, \alpha| := |z_1\alpha_2 - z_2\alpha_1|$$

is well-defined and continuous, where $z = (z_1 : z_2)$ and $\alpha = (\alpha_1 : \alpha_2)$ are unimodular coordinates.

Proof. Since a unimodular coordinate is unique up to a scalar multiple by $\mathcal{O}_L^\times$, we see that $|z,\alpha|$ is well-defined.

For the continuity, consider the natural map

$$g : L^2\backslash\{(0,0)\} \to \mathbb{P}^1(L).$$

Since the translation by the action of $L^\times$ on $L^2\backslash\{(0,0)\}$ is a homeomorphism, the continuous map g is open. Put

$$U_1 = \mathcal{O}_L^\times \times (\mathcal{O}_L\backslash\{0\}), \quad U_2 = (\mathcal{O}_L\backslash\{0\}) \times \mathcal{O}_L^\times.$$

For any $i = 1, 2$, the map

$$w_i : U_i \to \mathbb{R}_{\geqslant 0}, \quad (\alpha_1, \alpha_2) \mapsto |z_1\alpha_2 - z_2\alpha_1|$$

is continuous. For any open subset $U \subseteq \mathbb{R}_{\geqslant 0}$, we have

$$|z, -|^{-1}(U) = g(w_1^{-1}(U)) \cup g(w_2^{-1}(U)),$$

which is open. □

Definition 4.37. Let L be K_∞ or $\mathbb{C}_\infty$. For any $\alpha \in \mathbb{P}^1(L)$ and any $\rho \in q^{\mathbb{Q}}$, let

$$\mathscr{D}_L(\alpha, \rho) = \{z \in \mathbb{P}^1(L) \mid |z, \alpha| \leqslant \rho\},$$
$$\mathscr{D}_L^\circ(\alpha, \rho) = \{z \in \mathbb{P}^1(L) \mid |z, \alpha| < \rho\}.$$

We write $\mathscr{D}_{K_\infty}(\alpha, \rho) = \mathscr{D}(\alpha, \rho)$ and $\mathscr{D}_{K_\infty}^\circ(\alpha, \rho) = \mathscr{D}^\circ(\alpha, \rho)$.

Note that we have

$$\alpha \in \mathscr{D}_L^\circ(\alpha, \rho) \subseteq \mathscr{D}_L(\alpha, \rho).$$

Lemma 4.38. *Let $s \geqslant 0$ be any rational number. Let $\alpha \in \mathbb{P}^1(L)$ and let $\rho \in q^{\mathbb{Q}}$.*

(1) the disc $\mathscr{D}_L(\alpha, \rho)$ depends only on $r_{s,L}(\alpha)$ if $\rho \geqslant q^{-s}$.
(2) the disc $\mathscr{D}_L^\circ(\alpha, \rho)$ depends only on $r_{s,L}(\alpha)$ if $\rho > q^{-s}$.

Proof. Take any $\alpha, \beta \in \mathbb{P}^1(L)$ with unimodular coordinates $\alpha = (\alpha_1 : \alpha_2)$ and $\beta = (\beta_1 : \beta_2)$. Then $r_{s,L}(\alpha) = r_{s,L}(\beta)$ if and only if

$$\mathcal{O}_{L,s}(\alpha_1 f_1 + \alpha_2 f_2) = \mathcal{O}_{L,s}(\beta_1 f_1 + \beta_2 f_2).$$

This is the same as saying that there exists $c \in \mathcal{O}_L^\times$ satisfying

$$c\alpha_1 \equiv \beta_1, \quad c\alpha_2 \equiv \beta_2 \bmod m_L^{\geqslant s}.$$

From this it follows that for any unimodular coordinate $z = (z_1 : z_2)$ we have

$$|z_1\alpha_2 - z_2\alpha_1| \leqslant \rho \quad \Leftrightarrow \quad |z_1\beta_2 - z_2\beta_1| \leqslant \rho \quad (q^{-s} \leqslant \rho),$$
$$|z_1\alpha_2 - z_2\alpha_1| < \rho \quad \Leftrightarrow \quad |z_1\beta_2 - z_2\beta_1| < \rho \quad (q^{-s} < \rho).$$

This concludes the proof. □

Lemma 4.39. *For any $\alpha \in \mathbb{P}^1(L)$ and any positive rational number $s > 0$, we have*

$$\mathscr{D}_L(\alpha, q^{-s}) = \{z \in \mathbb{P}^1(L) \mid r_{s,L}(z) = r_{s,L}(\alpha)\}.$$

Proof. By Lemma 4.38, if $z \in \mathbb{P}^1(L)$ satisfies $r_{s,L}(z) = r_{s,L}(\alpha)$ then we have $z \in \mathscr{D}_L(z, q^{-s}) = \mathscr{D}_L(\alpha, q^{-s})$. Conversely, take any $z \in \mathscr{D}_L(\alpha, q^{-s})$. Let $z = (z_1 : z_2)$ and $\alpha = (\alpha_1 : \alpha_2)$ be unimodular coordinates. Then we have $z_1\alpha_2 - z_2\alpha_1 \in m_L^{\geqslant s}$. If $|\alpha_1| = 1$, then the assumption $s > 0$ implies $|z_1| = 1$ and

$$\begin{aligned}\mathcal{O}_{L,s}(z_1 f_1 + z_2 f_2) &= \mathcal{O}_{L,s}(\alpha_1 z_1 f_1 + \alpha_1 z_2 f_2)\\ &= \mathcal{O}_{L,s}(\alpha_1 z_1 f_1 + \alpha_2 z_1 f_2) = \mathcal{O}_{L,s}(\alpha_1 f_1 + \alpha_2 f_2),\end{aligned}$$

which shows $r_{s,L}(z) = r_{s,L}(\alpha)$. The case of $|\alpha_2| = 1$ can be treated similarly. □

Corollary 4.40. *Let $s > 0$ be any positive rational number. For any $z, z', z'' \in \mathbb{P}^1(L)$, we have*

$$|z, z'|,\ |z', z''| \leqslant q^{-s} \quad \Rightarrow \quad |z, z''| \leqslant q^{-s}.$$

Proof. Since $s > 0$, Lemma 4.39 and the assumption show

$$r_{s,L}(z) = r_{s,L}(z') = r_{s,L}(z''),$$

which yields $|z', z''| \leqslant q^{-s}$. □

Lemma 4.41. *Let $n \geqslant 0$ be any integer and let $\rho \in q^{\mathbb{Q}} \cap [q^{-n-1}, q^{-n})$. For any integer $i \in [0, n]$ and any $a \in K_\infty$ with $i = \max\{0, \log_q |a|\}$, we have*

$$D_L(a, q^{2i}\rho) = \mathscr{D}_L(\alpha, \rho), \quad \alpha = (1 : a).$$

Moreover, for any $\rho < 1$ we have $D_L(\infty, \rho) = \mathscr{D}_L(\infty, \rho)$.

Proof. If $i = 0$, then $\alpha = (1 : a)$ is a unimodular coordinate. Since $\rho < 1$, if $z = (z_1 : z_2)$ is a unimodular coordinate then

$$|z_1 a - z_2| \leqslant \rho \quad \Leftrightarrow \quad \left(|z_1| = 1 \text{ and } \left|\frac{z_2}{z_1} - a\right| \leqslant \rho\right).$$

Since $|a| \leqslant 1$, the condition $|\frac{z_2}{z_1} - a| \leqslant \rho$ implies $|\frac{z_2}{z_1}| \leqslant 1$ and $|z_1| = 1$. This yields the lemma for this case.

If $i \in [1, n]$, then $|a| = q^i$. Hence $\alpha = (\pi_\infty^i, \pi_\infty^i a)$ is a unimodular coordinate with $|\pi_\infty^i a| = 1$ and for any unimodular coordinate $z = (z_1 : z_2)$ we have

$$|z_1(\pi_\infty^i a) - z_2(\pi_\infty^i)| \leqslant \rho \quad \Leftrightarrow \quad \left(|z_1| = q^{-i} \text{ and } \left|\frac{z_2}{z_1} - a\right| \leqslant q^{2i}\rho\right).$$

In the latter condition $|z_1| = q^{-i}$ is superfluous, since $|a| = q^i$ and $\left|\frac{z_2}{z_1} - a\right| \leqslant q^{2i}\rho$ force $|\frac{z_2}{z_1}| = q^i$, which yields $|z_2| = 1$ and $|z_1| = q^{-i}$. Thus the lemma follows also for this case.

Finally, for $\infty = (0:1)$ we have

$$|z_1| \leqslant \rho \quad \Leftrightarrow \quad \left(|z_2| = 1 \text{ and } \left|\frac{z_1}{z_2}\right| \leqslant \rho\right),$$

which yields the last assertion. □

Corollary 4.42. *Let $n \geqslant 0$ be an integer and let Λ be a complete set of representatives modulo π_∞^{n+1}. For any $e \in \mathcal{T}_1^o(n)$, we have*

$$U(e) = \mathscr{D}(c_\Lambda(e), q^{-n-1}), \quad \mathcal{U}(e) = \mathscr{D}_{\mathbb{C}_\infty}(c_\Lambda(e), q^{-n-1}).$$

Proof. Write $c_\Lambda(e) = (x_1 : x_2)$ with a unimodular coordinate. As we saw in the proof of Lemma 4.31, the direct summand $N = \mathcal{O}_\infty(x_1 f_1 + x_2 f_2)$ of M_0 corresponds to a half-line starting v_0 and passing through e. In particular, we have $t(e) = [N + \pi_\infty^{n+1} M_0]$.

Then $x' = (x_1' : x_2') \in \mathbb{P}^1(K_\infty)$ of unimodular coordinate lies in $U(e)$ if and only if the half-line starting from v_0 corresponding to the direct summand $N' = \mathcal{O}_\infty(x_1' f_1 + x_2' f_2)$ passes through $t(e)$, that is $N + \pi_\infty^{n+1} M_0 = N' + \pi_\infty^{n+1} M_0$. This is the same as saying $r_{n+1}(x) = r_{n+1}(x')$. By Lemma 4.39, this is equivalent to $x' \in \mathscr{D}(x, q^{-n-1})$, and we obtain the first equality of the corollary.

By Lemma 4.33 and Lemma 4.41, we see that $\mathscr{D}(c_\Lambda(e), q^{-n-1})$ is an element of $\mathrm{DCD}(K_\infty)$ and its extension over $\mathbb{C}_\infty$ is $\mathscr{D}_{\mathbb{C}_\infty}(c_\Lambda(e), q^{-n-1})$. This yields the second equality. □

5. Drinfeld upper half plane

We call

$$\Omega = \mathbb{P}^1(\mathbb{C}_\infty) \backslash \mathbb{P}^1(K_\infty) = \mathbb{C}_\infty \backslash K_\infty$$

the Drinfeld upper half plane.

5.1. Coverings of Ω associated with vertices.

Definition 5.1. For any $v \in \mathcal{T}_0$, define

$$\mathcal{U}(v) = \mathbb{P}^1(\mathbb{C}_\infty) \backslash \coprod_{o(e)=v} \mathcal{U}(e).$$

Example 5.2. By Example 4.21, we can write

$$\begin{aligned}\mathcal{U}(v_0) &= \mathbb{P}^1(\mathbb{C}_\infty) \backslash \left(\coprod_{\lambda \in \mathbb{F}_q} D_{\mathbb{C}_\infty}(\lambda, q^{-1}) \sqcup D'_{\mathbb{C}_\infty}(0, q) \right) \\ &= \{ z \in \mathbb{C}_\infty \mid q^{-1} < |z| < q, \quad |z - \lambda| > q^{-1} \text{ for all } \lambda \in \mathbb{F}_q^\times \}.\end{aligned}$$

By (4.7), we have

$$\gamma(\mathcal{U}(v)) = \mathcal{U}(\gamma \circ v) \quad \text{for any } \gamma \in GL_2(K_\infty).$$

Lemma 5.3.

$$\{\gamma \in GL_2(K_\infty) \mid \gamma(\mathcal{U}(v_0)) = \mathcal{U}(v_0)\} = GL_2(\mathcal{O}_\infty)K_\infty^\times.$$

Proof. Take any $\gamma \in GL_2(K_\infty)$. Then γ stabilizes $\mathcal{U}(v_0)$ if and only if it stabilizes $\coprod_{\lambda \in \mathbb{F}_q} D_{\mathbb{C}_\infty}(\lambda, q^{-1}) \sqcup D'_{\mathbb{C}_\infty}(0, q)$. Since γ acts on $\mathrm{DCD}(\mathbb{C}_\infty)$, this is the same as saying that γ stabilizes the subset S_{v_0} of $\mathrm{DCD}(\mathbb{C}_\infty)$ consisting of these discs.

Let $\gamma_\lambda = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}$. By Lemma 4.8, we have

$$\begin{aligned} D_{\mathbb{C}_\infty}(\lambda, q^{-1}) &= \begin{pmatrix} \pi_\infty & \lambda \\ 0 & 1 \end{pmatrix} (D_{\mathbb{C}_\infty}(0,1)) \\ &= \begin{pmatrix} \pi_\infty & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \pi_\infty & 0 \end{pmatrix} (D'_{\mathbb{C}_\infty}(0,q)) = \gamma_\lambda(D'_{\mathbb{C}_\infty}(0,q)). \end{aligned}$$

By Corollary 4.12, we have the identification

$$S_{v_0} = \left(\left(\coprod_{\lambda \in \mathbb{F}_q} \gamma_\lambda K_0(\pi_\infty) \sqcup K_0(\pi_\infty) \right) K_\infty^\times \right) / K_0(\pi_\infty) K_\infty^\times.$$

Now the bijection

$$GL_2(\mathcal{O}_\infty)/K_0(\pi_\infty) \to \mathbb{P}^1(\mathbb{F}_q), \quad \gamma \mapsto \gamma \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

yields $GL_2(\mathcal{O}_\infty) = \coprod_{\lambda \in \mathbb{F}_q} \gamma_\lambda K_0(\pi_\infty) \sqcup K_0(\pi_\infty)$ and we obtain

$$S_{v_0} = GL_2(\mathcal{O}_\infty)K_\infty^\times / K_0(\pi_\infty)K_\infty^\times.$$

Thus the stabilizer agrees with $GL_2(\mathcal{O}_\infty)K_\infty^\times$. □

Lemma 5.4.

$$\mathcal{U}(v) \cap \mathbb{P}^1(K_\infty) = \varnothing.$$

Proof. Translating by the action of $GL_2(K_\infty)$, we may assume $v = v_0$. Take any $x \in \mathcal{U}(v_0) \cap \mathbb{P}^1(K_\infty)$. Then $x \in K_\infty$. Since $q^{-1} < |x| < q$, we have $|x| = 1$. Write $x = \sum_{i \geqslant 0} a_i \pi_\infty^i$ with $a_i \in \mathbb{F}_q$ and $a_0 \neq 0$. Then we have $|x - a_0| \leqslant q^{-1}$, which is a contradiction. □

Lemma 5.5. *Let $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$.*

1. *If $D(a,\rho) \supsetneq D(a',\rho')$, then $|a - a'| \leqslant \rho$ and $\rho \geqslant q\rho'$.*
2. *If $D'(a,\rho) \supsetneq D(a',\rho')$, then $|a - a'| \geqslant \max\{\rho, q\rho'\}$.*
3. *If $D'(a,\rho) \supsetneq D'(a',\rho')$, then $|a - a'| \leqslant q^{-1}\rho'$ and $\rho' \geqslant q\rho$.*

Proof. For (1), the assumption yields $|a-a'| \leqslant \rho$ and $\rho > \rho'$, since $\rho \leqslant \rho'$ would imply $D(a,\rho) \subseteq D(a',\rho')$. Since $\rho, \rho' \in q^{\mathbb{Z}}$, this forces $\rho \geqslant q\rho'$.

For (2), the assumption means

$$D^\circ(a,\rho) \cap D(a',\rho') = D(a, q^{-1}\rho) \cap D(a',\rho') = \varnothing$$

and thus $|a-a'| > \max\{q^{-1}\rho, \rho'\}$, which forces $|a-a'| \geqslant \max\{\rho, q\rho'\}$.

For (3), the assumption means $D(a, q^{-1}\rho) \subsetneq D(a', q^{-1}\rho')$ and (1) concludes the proof. □

Lemma 5.6. *If $e, e' \in \mathcal{T}_1^o$ satisfy $U(e') \subsetneq U(e)$, then we have*

$$\mathbb{P}^1(\mathbb{C}_\infty) \backslash \mathcal{U}(-e') \subseteq \mathcal{U}(e).$$

Proof. Suppose $U(e) = D(a,\rho)$ and $U(e') = D(a',\rho')$ with some $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$. Then Lemma 5.5 (1) yields $|a-a'| \leqslant \rho$ and $\rho \geqslant q\rho'$. Since $U(-e') = D'(a', q\rho')$, for any $z \in \mathbb{P}^1(\mathbb{C}_\infty)\backslash\mathcal{U}(-e')$ we have $|z-a'| < q\rho'$ and

$$|z-a| = |z-a' + (a'-a)| \leqslant \max\{q\rho', \rho\} = \rho,$$

which gives $z \in \mathcal{U}(e)$.

Suppose $U(e) = D'(a,\rho)$ and $U(e') = D(a',\rho')$ with some $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$. Then Lemma 5.5 (2) yields $|a-a'| \geqslant \max\{\rho, q\rho'\}$. Since $U(-e') = D'(a', q\rho')$, for any $z \in \mathbb{P}^1(\mathbb{C}_\infty)\backslash\mathcal{U}(-e')$ we have $|z-a'| < q\rho'$ and

$$|z-a| = |z-a'+(a'-a)| = |a'-a| \geqslant \max\{\rho, q\rho'\} \geqslant \rho.$$

Hence $z \in \mathcal{U}(e)$.

Suppose $U(e) = D'(a,\rho)$ and $U(e') = D'(a',\rho')$ with some $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$. Then Lemma 5.5 (3) yields $|a-a'| \leqslant q^{-1}\rho'$ and $\rho' \geqslant q\rho$. Since $U(-e') = D(a', q^{-1}\rho')$, for any $z \in \mathbb{P}^1(\mathbb{C}_\infty)\backslash\mathcal{U}(-e')$ we have $|z-a'| > q^{-1}\rho'$ and

$$|z-a| = |z-a'+(a'-a)| = |z-a'| > q^{-1}\rho' \geqslant \rho,$$

which gives $z \in \mathcal{U}(e)$. This concludes the proof. □

Definition 5.7. For any $e \in \mathcal{T}_1^o$, put

$$\mathcal{U}^\star(e) := \begin{cases} D^\circ_{\mathbb{C}_\infty}(a, q^{\frac{1}{3}}\rho) & (U(e) = D(a,\rho)), \\ D'^\circ_{\mathbb{C}_\infty}(a, q^{-\frac{1}{3}}\rho) & (U(e) = D'(a,\rho)). \end{cases}$$

Then we have $\mathcal{U}(e) \subseteq \mathcal{U}^\star(e)$ and $\mathbb{P}^1(K_\infty) \cap \mathcal{U}^\star(e) = \mathbb{P}^1(K_\infty) \cap \mathcal{U}(e)$.

Lemma 5.8. *$\mathcal{U}^\star(e)$ is independent of the choice of a center a of $U(e)$.*

Proof. Suppose $U(e) = D(a, \rho) = D(b, \rho)$ with some $a, b \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$, so that $|a - b| \leqslant \rho$. Then we have $|a - b| < q^{\frac{1}{3}}\rho$ and $D^\circ_{\mathbb{C}_\infty}(a, q^{\frac{1}{3}}\rho) = D^\circ_{\mathbb{C}_\infty}(b, q^{\frac{1}{3}}\rho)$.

Suppose $U(e) = D'(a, \rho) = D'(b, \rho)$ with some $a, b \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$, so that $|a - b| < \rho$ and thus $|a-b| \leqslant q^{-1}\rho$. Then we have $|a-b| < q^{-\frac{1}{3}}\rho$ and $D'^\circ_{\mathbb{C}_\infty}(a, q^{-\frac{1}{3}}\rho) = D'^\circ_{\mathbb{C}_\infty}(b, q^{-\frac{1}{3}}\rho)$. This concludes the proof. ☐

Lemma 5.9. *If $e, e' \in \mathcal{T}_1^o$ satisfy $U(e) \cap U(e') = \varnothing$, then $\mathcal{U}^\star(e) \cap \mathcal{U}^\star(e') = \varnothing$.*

Proof. Since $\infty \in D'(a, \rho)$, we may assume $U(e) = D(a, \rho)$ with some $a \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$.

Suppose $U(e') = D(a', \rho')$ with some $a' \in K_\infty$ and $\rho' \in q^{\mathbb{Z}}$. If $\mathcal{U}^\star(e) \cap \mathcal{U}^\star(e') \neq \varnothing$, then for some $z \in \mathbb{C}_\infty$ we have

$$|z - a| \leqslant q^{\frac{1}{3}}\rho, \quad |z - a'| \leqslant q^{\frac{1}{3}}\rho',$$

so that

$$|a - a'| = |(z - a') - (z - a)| \leqslant q^{\frac{1}{3}} \max\{\rho, \rho'\}.$$

Since $a, a' \in K_\infty$, it forces $|a - a'| \leqslant \max\{\rho, \rho'\}$ and thus $D(a, \rho) \cap D(a', \rho') \neq \varnothing$, which is a contradiction.

Suppose $U(e') = D'(a', \rho')$ with some $a' \in K_\infty$ and $\rho' \in q^{\mathbb{Z}}$. Then the assumption $U(e) \cap U(e') = \varnothing$ yields

$$D(a, \rho) \subseteq D^\circ(a', \rho') = D(a', q^{-1}\rho'),$$

which implies $|a - a'| \leqslant q^{-1}\rho'$. Moreover, for any $\varpi_\rho \in K_\infty$ with $|\varpi_\rho| = \rho$, we have $|\varpi_\rho + a - a'| \leqslant q^{-1}\rho'$ and thus $\rho \leqslant q^{-1}\rho'$.

If $z \in \mathbb{C}_\infty$ satisfies $|z - a| \leqslant q^{\frac{1}{3}}\rho$, then we have

$$|z - a'| = |z - a + (a - a')| \leqslant \max\{q^{\frac{1}{3}}\rho, q^{-1}\rho'\} < q^{-\frac{1}{3}}\rho',$$

which yields $\mathcal{U}^\star(e) \cap \mathcal{U}^\star(e') = \varnothing$. This concludes the proof. ☐

Lemma 5.10. *If $e \neq e' \in \mathcal{T}_1^o$ satisfy $U(e) \supsetneq U(e')$, then $\mathcal{U}(e) \supseteq \mathcal{U}^\star(e')$.*

Proof. First suppose $U(e) = D(a, \rho)$ and $U(e') = D(a', \rho')$ for some $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$. Then Lemma 5.5 (1) yields $|a - a'| \leqslant \rho$ and $\rho \geqslant q\rho'$. If $z \in \mathbb{C}_\infty$ satisfies $|z - a'| \leqslant q^{\frac{1}{3}}\rho'$, then we have

$$|z - a| = |z - a' + (a' - a)| \leqslant \max\{q^{\frac{1}{3}}\rho', \rho\} = \rho$$

and the lemma follows for this case.

Suppose $U(e) = D'(a, \rho)$ and $U(e') = D(a', \rho')$ for some $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$. Then Lemma 5.5 (2) yields $|a - a'| \geqslant \max\{\rho, q\rho'\}$. If $z \in \mathbb{C}_\infty$ satisfies $|z - a'| \leqslant q^{\frac{1}{3}}\rho'$, then the inequality $q^{\frac{1}{3}}\rho' < q\rho'$ yields

$$|z - a| = |z - a' + (a' - a)| = |a' - a| \geqslant \rho$$

and the lemma follows for this case.

Finally, suppose $U(e) = D'(a,\rho)$ and $U(e') = D'(a',\rho')$ for some $a, a' \in K_\infty$ and $\rho, \rho' \in q^{\mathbb{Z}}$. Then Lemma 5.5 (3) yields $|a - a'| \leqslant q^{-1}\rho'$ and $\rho' \geqslant q\rho$. If $z \in \mathbb{C}_\infty$ satisfies $|z - a'| \geqslant q^{-\frac{1}{3}}\rho'$, then the inequality $q^{-1}\rho' < q^{\frac{-1}{3}}\rho'$ yields

$$|z-a| = |z - a' + (a' - a)| = |z - a'| \geqslant q^{-\frac{1}{3}}\rho' \geqslant q^{\frac{2}{3}}\rho > \rho,$$

and the lemma also follows for this case. □

Definition 5.11. For any $v \in \mathcal{T}_0$, we define

$$\mathcal{U}^\star(v) := \mathbb{P}^1(\mathbb{C}_\infty) \backslash \left(\coprod_{o(e)=v} \mathcal{U}^\star(e) \right),$$

where the union on the right-hand side is disjoint by Lemma 5.9. By Lemma 5.4, we have

$$\mathcal{U}^\star(v) \subseteq \mathcal{U}(v), \quad \mathcal{U}^\star(v) \cap \mathbb{P}^1(K_\infty) = \mathcal{U}(v) \cap \mathbb{P}^1(K_\infty) = \varnothing. \tag{5.1}$$

Lemma 5.12.

$$\Omega = \bigcup_{v \in \mathcal{T}_0} \mathcal{U}(v) = \bigcup_{v \in \mathcal{T}_0} \mathcal{U}^\star(v).$$

Proof. By (5.1), it is enough to show $\Omega \subseteq \bigcup_{v \in \mathcal{T}_0} \mathcal{U}^\star(v)$.

Take any $z \in \Omega$ and suppose $z \notin \mathcal{U}^\star(v)$ for any $v \in \mathcal{T}_0$. This means that for any $v \in \mathcal{T}_0$ there exists $e \in \mathcal{T}_1^o$ with $o(e) = v$ satisfying $z \in \mathcal{U}^\star(e)$.

Fix $w_0 \in \mathcal{T}_0$. Then we can find $e'_0 = (w_0 \to w_1) \in \mathcal{T}_1^o$ satisfying $z \in \mathcal{U}^\star(e'_0)$. Similarly, we can find $e'_1 = (w_1 \to w_2) \in \mathcal{T}_1^o$ satisfying $z \in \mathcal{U}^\star(e'_1)$. Since Lemma 5.9 implies $\mathcal{U}^\star(e'_0) \cap \mathcal{U}^\star(-e'_0) = \varnothing$, we have $w_2 \neq w_0$. Repeating this, we can find a half-line $H = \{w_i\}_{i \geqslant 0}$ in $\mathcal{T}$ satisfying $z \in \mathcal{U}^\star(e'_i)$ with $e'_i = (w_i \to w_{i+1})$ for any i.

Since $U(e'_i) \supsetneq U(e'_{i+1})$, Lemma 5.10 yields

$$\mathcal{U}(e'_i) \supseteq \mathcal{U}^\star(e'_{i+1}) \supseteq \mathcal{U}(e'_{i+1})$$

for any $i \geqslant 0$. By Lemma 4.23, we obtain

$$z \in \bigcap_{i \geqslant 1} \mathcal{U}^\star(e'_i) \subseteq \bigcap_{i \geqslant 0} \mathcal{U}(e'_i) = \{\lim(H)\} \subseteq \mathbb{P}^1(K_\infty),$$

which is a contradiction. □

5.2. **Annuli in $\mathbb{P}^1(\mathbb{C}_\infty)$ associated with edges.**

Definition 5.13. For any $e \in \mathcal{T}_1^o$, define

$$\mathcal{V}(e) = \mathcal{U}(o(e)) \cap \mathcal{U}(t(e)).$$

Note $\mathcal{V}(e) = \mathcal{V}(-e)$. By (4.6) and Definition 5.1, we obtain

$$\mathcal{V}(e) = \mathbb{P}^1(\mathbb{C}_\infty) \backslash (\mathcal{U}(e) \sqcup \mathcal{U}(-e)).$$

By (4.4), either of $U(e)$ and $U(-e)$ equals $D(a, \rho)$ with some $a \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$, and the other is its complement $D'(a, q\rho)$ in $\mathbb{P}^1(K_\infty)$. Hence we have

$$\{\mathcal{U}(e), \mathcal{U}(-e)\} = \{D_{\mathbb{C}_\infty}(a, \rho), D'_{\mathbb{C}_\infty}(a, q\rho)\}.$$

Thus we obtain

$$\mathcal{V}(e) = \{z \in \mathbb{C}_\infty \mid \rho < |z - a| < q\rho\} \subseteq \Omega.$$

In particular, $\mathcal{V}(e)$ is an open annulus defined over K_∞.

Example 5.14. By Example 4.22, we have

$$\mathcal{V}(e_0) = \{z \in \mathbb{C}_\infty \mid 1 < |z| < q\}.$$

Lemma 5.15. *For any $v, v' \in \mathcal{T}_0$, we have*

$$\mathcal{U}(v) \cap \mathcal{U}(v') \neq \varnothing \quad \Leftrightarrow \quad d(v, v') \leqslant 1,$$

in which case $\mathcal{U}(v) = \mathcal{U}(v')$ if $v = v'$ and $\mathcal{U}(v) \cap \mathcal{U}(v') = \mathcal{V}(e) = \mathcal{V}(-e)$ with $e = (v \to v')$ otherwise.

Proof. It is enough to show $\mathcal{U}(v) \cap \mathcal{U}(v') = \varnothing$ if $d(v, v') \geqslant 2$. Translating by the action of $GL_2(K_\infty)$, we may assume $v = v_0$ and $v' = \gamma \circ v_0$ with some $\gamma \in GL_2(K_\infty)$. By the elementary divisor theorem and (2.2), we may assume $\gamma = \begin{pmatrix} \pi_\infty^m & 0 \\ 0 & \pi_\infty^n \end{pmatrix}$ with some integers $m, n \geqslant 0$. If $z \in \mathcal{U}(v_0)$ and $\gamma(z) \in \mathcal{U}(v_0)$, then we have

$$q^{-1} < |z| < q \quad \text{and} \quad q^{-1} < |\pi_\infty^{m-n} z| < q,$$

which occurs only if $-1 \leqslant m - n \leqslant 1$, namely $d(v, v') \leqslant 1$. This concludes the proof. □

5.3. **Irrational absolute value.**

Definition 5.16. For any $z \in \mathbb{C}_\infty$, define the irrational absolute value of z by

$$|z|_i = \inf_{a \in K_\infty} |z - a|.$$

Note that for any $z \in \mathbb{C}_\infty$ and $a \in K_\infty$ we have

$$|a| > |z| \Rightarrow |z - a| = |a| > |z|, \quad |a| < |z| \Rightarrow |z - a| = |z|. \tag{5.2}$$

Lemma 5.17. *For any $z \in \mathbb{C}_\infty$, we have $|z|_i = |z-a|$ for some $a \in K_\infty$.*

Proof. From (5.2), it follows that $|z|_i$ is the infimum of $|z-a|$ on the compact set $\{a \in K_\infty \mid |a| \leqslant |z|\}$, which is attained by some a in this set. ☐

Lemma 5.18 ([DH], Proposition 5.2)**.** *Let $z \in \mathbb{C}_\infty$.*

(1) $|z|_i = 0$ if and only if $z \in K_\infty$.
(2) For any $c \in K_\infty$, we have $|cz|_i = |c||z|_i$.
(3) If $|z| \notin q^{\mathbb{Z}}$, then $|z|_i = |z|$.
(4) Suppose $|z| = 1$ and let $\bar{z} \in \bar{\mathbb{F}}_q$ be the residue class of z. Then $|z|_i = |z| = 1$ if and only if $\bar{z} \notin \mathbb{F}_q$.

Proof. The first assertion follows from Lemma 5.17, and the second assertions is clear. For the third, if $|z| \notin q^{\mathbb{Z}}$ then $|z| \neq |a|$ for any $a \in K_\infty$. Thus we have $|z-a| = \max\{|z|,|a|\} \geqslant |z|$ and the equality holds for $a = 0$. This implies $|z|_i = |z|$. For the fourth, take any $a \in K_\infty$. If $|z| = |a|$, then $|z-a| < |z| = 1$ if and only if $\bar{z} = \bar{a} \in \mathbb{F}_q$. Combined with (5.2), this shows the fourth assertion. ☐

Definition 5.19. For any $r, s \in \mathbb{Q}$, let

$$\Omega_r = \{z \in \mathbb{C}_\infty \mid |z|_i \geqslant q^{-r}\}, \quad \Omega_{r,s} = \{z \in \Omega_r \mid |z| \leqslant q^s\}.$$

Then Lemma 5.18 (1) shows $\Omega_{r,s} \subseteq \Omega_r \subseteq \Omega$ and

$$\Omega_r = \bigcup_{s \in \mathbb{Z}_{\geqslant 0}} \Omega_{r,s}, \quad \Omega = \bigcup_{r \in \mathbb{Z}_{\geqslant 0}} \Omega_r = \bigcup_{r,s \in \mathbb{Z}_{\geqslant 0}} \Omega_{r,s}.$$

Lemma 5.20. *For any rational numbers r, s with $s \geqslant -r$, there exists a finite set $J \subseteq K_\infty$ satisfying*

$$\begin{cases} \mathbb{P}^1(K_\infty) = D^\circ(\infty, q^{-s}) \sqcup \coprod_{a \in J} D^\circ(a, q^{-r}), \\ \Omega_{r,s} = \mathbb{P}^1(\mathbb{C}_\infty) \backslash \left(D^\circ_{\mathbb{C}_\infty}(\infty, q^{-s}) \sqcup \coprod_{a \in J} D^\circ_{\mathbb{C}_\infty}(a, q^{-r}) \right). \end{cases}$$

Proof. Since $q^{-(r+1)} < q^{-r} \leqslant q^s$, for any $a \in D(0, q^s)$ Lemma 4.2 yields $D(a, q^{-(r+1)}) \subseteq D(0, q^s)$. Thus we can find a finite set $J \subseteq K_\infty$ satisfying

$$\mathbb{P}^1(K_\infty) = D^\circ(\infty, q^{-s}) \sqcup \coprod_{a \in J} D^\circ(a, q^{-r}).$$

This means that for any $b \in K_\infty$ we have $|b| > q^s$ or $|a-b| < q^{-r}$ for some $a \in J$.

Since the union in the first equality of the lemma is disjoint, we have $|a| \leqslant q^s$ and $|a-a'| \geqslant q^{-r}$ for any $a, a' \in J$. Since $s \geqslant -r$, this implies that the union in the second equality of the lemma is also disjoint.

We show the latter equality in the lemma. It is clear that $\Omega_{r,s}$ is contained in the set on the right-hand side. Conversely, take any $z \in \mathbb{C}_\infty$ satisfying $|z| \leqslant q^s$ and $|z-a| \geqslant q^{-r}$ for any $a \in J$. Suppose $|z|_i < q^{-r}$. Then we have $|z-b| < q^{-r}$ for some $b \in K_\infty$.

If $|b| > q^s$, then we have $|z-b| = |b| > q^s \geqslant q^{-r}$, which contradicts $|z-b| < q^{-r}$. Thus we obtain $|a-b| < q^{-r}$ for some $a \in J$ and $|z-a| < q^{-r}$, which is a contradiction. This shows $|z|_i \geqslant q^{-r}$ and $z \in \Omega_{r,s}$. □

5.4. **Rigid analytic structure of Ω.**

Lemma 5.21. *Let $\alpha \in \mathbb{P}^1(K_\infty)$ and let Y be any affinoid variety over $\mathbb{C}_\infty$. Let $\varphi : Y \to \mathbb{P}^1_{\mathbb{C}_\infty}\backslash\{\alpha\}$ be any morphism of rigid analytic varieties over $\mathbb{C}_\infty$. Then we have*

$$\varphi(Y) \subseteq \mathbb{P}^1_{\mathbb{C}_\infty}\backslash D_{\mathbb{C}_\infty}(\alpha, q^{-m})$$

for some integer m.

Proof. First suppose $\alpha = \infty$, so that

$$D_{\mathbb{C}_\infty}(\alpha, q^{-m}) = \{z \in \mathbb{C}_\infty \mid |z| \geqslant q^m\} \cup \{\infty\}.$$

Put $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}_\infty[T])$ so that $\mathbb{P}^1_{\mathbb{C}_\infty}\backslash\{\infty\}$ is its analytification. By the maximal modulus principle on Y for the function $\varphi^*(T)$, there exists a positive rational number $s > 0$ satisfying

$$|\varphi(y)| \leqslant q^s \quad \text{for any } y \in Y.$$

Then any integer m satisfying $s < m$ has the desired property.

Suppose $\alpha \neq \infty$. We can find $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K_\infty)$ satisfying $\gamma(\alpha) = \infty$. Then $c\alpha + d = 0$. For any $\rho \in q^{\mathbb{Q}}$ satisfying $\rho|ac| < 1$, Lemma 4.8 yields

$$\gamma(D_{\mathbb{C}_\infty}(\alpha, \rho)) = D'_{\mathbb{C}_\infty}\left(\frac{a}{c}, \frac{1}{\rho|c|^2}\right) = D'_{\mathbb{C}_\infty}\left(0, \frac{1}{\rho|c|^2}\right) = D_{\mathbb{C}_\infty}\left(\infty, \rho|c|^2\right).$$

Applying the lemma for $\alpha = \infty$ to the morphism

$$\gamma \circ \varphi : Y \to \mathbb{P}^1_{\mathbb{C}_\infty}\backslash\{\alpha\} \to \mathbb{P}^1_{\mathbb{C}_\infty}\backslash\{\infty\},$$

we can find an integer m' satisfying $\gamma(\varphi(Y)) \cap D_{\mathbb{C}_\infty}(\infty, q^{-m'}) = \varnothing$. Then the lemma holds for any integer m satisfying $q^m \geqslant \max\{q^{m'}|c|^2, |ac|\}$. □

Lemma 5.22. *For any $r, s \in \mathbb{Z}$, the subset $\Omega_{r,s}$ is an admissible affinoid open subset of $\mathbb{P}^1_{\mathbb{C}_\infty}$, and Ω_r is an admissible open subset of $\mathbb{P}^1_{\mathbb{C}_\infty}$. Moreover, $\{\Omega_{r,s}\}_{s \in \mathbb{Z}}$ is an admissible open covering of Ω_r.*

Proof. Since $\Omega_{r,s} \subseteq \Omega_r \subseteq \mathbb{A}^1_{\mathbb{C}_\infty}$ and $\mathbb{A}^1_{\mathbb{C}_\infty}$ is an open subvariety of $\mathbb{P}^1_{\mathbb{C}_\infty}$, it is enough to show that $\Omega_{r,s}$ and Ω_r are admissible open subsets of $\mathbb{A}^1_{\mathbb{C}_\infty}$.

Note that $\{D_{\mathbb{C}_\infty}(0, q^s)\}_{s \in \mathbb{Z}_{\geqslant 0}}$ is an admissible open covering of $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_\infty}$ and $\Omega_{r,s} = \Omega_r \cap D_{\mathbb{C}_\infty}(0, q^s)$. Since $\Omega_{r,s}$ is a rational subdomain of $D_{\mathbb{C}_\infty}(0, q^s)$, it is an affinoid variety over $\mathbb{C}_\infty$. The property (G_2) of [BGR, §9.1.2] implies that Ω_r is an admissible open subset of $\mathbb{A}^1_{\mathbb{C}_\infty}$, and thus so is $\Omega_{r,s}$. The last assertion follows from [BGR, Definition 9.1.1/1 (iv)]. □

Lemma 5.23. *Let Y be an affinoid variety over $\mathbb{C}_\infty$ and let $\varphi : Y \to \mathbb{P}^1_{\mathbb{C}_\infty}$ be any morphism of rigid analytic varieties over $\mathbb{C}_\infty$ satisfying $\varphi(Y) \subseteq \Omega$. Then φ factors through $\Omega_{r,s}$ with some integers $r, s \geqslant 0$.*

Proof. By Lemma 5.21, for any $b \in \mathbb{P}^1(K_\infty)$ we can find a positive integer n_b satisfying

$$\varphi(Y) \subseteq \mathbb{P}^1_{\mathbb{C}_\infty} \backslash D_{\mathbb{C}_\infty}(b, q^{-n_b}).$$

Since $\mathbb{P}^1(K_\infty)$ is compact, there exists a finite subset I of $\mathbb{P}^1(K_\infty)$ satisfying

$$\mathbb{P}^1(K_\infty) = \bigcup_{b \in I} D(b, q^{-n_b}).$$

In particular, we have $\infty \in I$.

Let $n = \max\{n_b \mid b \in I\} > 0$ and let $J_{n,n}$ be the finite subset of K_∞ as in Lemma 5.20 for $(r, s) = (n, n)$. For any $a \in J_{n,n} \cup \{\infty\}$, we have $a \in D(b, q^{-n_b})$ for some $b \in I$. Then

$$\begin{cases} |a - b| \leqslant q^{-n_b}, \quad D^\circ_{\mathbb{C}_\infty}(a, q^{-n}) \subseteq D_{\mathbb{C}_\infty}(b, q^{-n_b}) & (a, b \neq \infty), \\ |a| \geqslant q^{n_\infty}, \quad D^\circ_{\mathbb{C}_\infty}(a, q^{-n}) \subseteq D_{\mathbb{C}_\infty}(\infty, q^{-n_\infty}) & (a \neq \infty,\ b = \infty), \\ D^\circ_{\mathbb{C}_\infty}(\infty, q^{-n}) \subseteq D_{\mathbb{C}_\infty}(\infty, q^{-n_\infty}) & (a, b = \infty). \end{cases}$$

This yields

$$\varphi(Y) \subseteq \bigcap_{b \in I} \left(\mathbb{P}^1_{\mathbb{C}_\infty} \backslash D_{\mathbb{C}_\infty}(b, q^{-n_b})\right) \subseteq \Omega_{n,n},$$

which concludes the proof. □

Proposition 5.24. *The subset Ω is an admissible open subset of $\mathbb{P}^1_{\mathbb{C}_\infty}$. Moreover,*

$$\{\Omega_r\}_{r \in \mathbb{Z}_{\geqslant 0}}, \quad \{\Omega_{r,s}\}_{r,s \in \mathbb{Z}_{\geqslant 0}}$$

are admissible open coverings of Ω.

Proof. The definition of $\mathbb{P}^1_{\mathbb{C}_\infty}$ [BGR, Example 9.3.4/3] shows that

$$\mathbb{P}^1_{\mathbb{C}_\infty} = \mathrm{Sp}(\mathbb{C}_\infty\langle z\rangle) \cup \mathrm{Sp}(\mathbb{C}_\infty\langle w\rangle), \quad w = 1/z$$

is an admissible open covering. Let D be one of these closed unit discs.

To show the proposition, by combining [BGR, Definition 9.1.1/1 (iii)(iv)] with the properties (G_1) and (G_2) of [BGR, §9.1.2], it is enough to show that $D \cap \Omega$ is an admissible open subset of D and $\{D \cap \Omega_{r,s}\}_{r,s \in \mathbb{Z}_{\geqslant 0}}$ is an admissible open covering of $D \cap \Omega$. Note that $D \cap \Omega_{r,s}$ is a rational subdomain of D.

For this, consider a morphism $\varphi : Y \to D$ of affinoid varieties over $\mathbb{C}_\infty$ satisfying $\varphi(Y) \subseteq D \cap \Omega$. By Lemma 5.23, there exist integers $r, s \geqslant 0$ such that $\varphi(Y) \subseteq D \cap \Omega_{r,s}$. Then [BGR, Proposition 9.1.4/2 (i)] implies that $D \cap \Omega$ is an admissible open subset of D, and combined with this, [BGR, Proposition 9.1.4/2 (ii)] shows that $\{D \cap \Omega_{r,s}\}_{r,s \in \mathbb{Z}_{\geqslant 0}}$ is an admissible open covering of $D \cap \Omega$. □

Remark 5.25. Proposition 5.24 and the property (G_2) of [BGR, §9.1.2] imply that

$$\{\Omega_r\}_{r \in \mathbb{Z}}, \quad \{\Omega_{r,s}\}_{r,s \in \mathbb{Z}}$$

are also admissible open coverings of Ω, since the coverings of the proposition gives refinements of them.

Lemma 5.26. *Let $Q(X)$ be an element of $\mathbb{C}_\infty(X)$ without poles in Ω. Then the function*

$$\Omega \to \mathbb{C}_\infty, \quad z \mapsto Q(z)$$

is an element of $\mathcal{O}(\Omega)$.

Proof. Let S be the set of poles of $Q(X)$. Then S is finite and the scheme $\mathbb{P}^1_{\mathbb{C}_\infty} \backslash S$ is locally of finite type over $\mathbb{C}_\infty$. From [Con, Theorem 5.2.1.1], we see that the analytification $(\mathbb{P}^1_{\mathbb{C}_\infty} \backslash S)^{\mathrm{an}}$ is an open subvariety of $(\mathbb{P}^1_{\mathbb{C}_\infty})^{\mathrm{an}}$ which contains Ω as an open subvariety. Since $Q(z)$ is a rational function on $\mathbb{P}^1_{\mathbb{C}_\infty} \backslash S$, it defines a rigid analytic function on $(\mathbb{P}^1_{\mathbb{C}_\infty})^{\mathrm{an}}$ and thus that on Ω. □

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_\infty)$. Since the action of γ on $\mathbb{P}^1(\mathbb{C}_\infty)$ preserves $\mathbb{P}^1(K_\infty)$, we have a bijection

$$\gamma : \Omega \to \Omega, \quad z \mapsto \frac{az+b}{cz+d}.$$

Corollary 5.27. *The map $\gamma : \Omega \to \Omega$ is a morphism of rigid analytic varieties over $\mathbb{C}_\infty$.*

Proof. Since the map $\gamma : \mathbb{P}^1(\mathbb{C}_\infty) \to \mathbb{P}^1(\mathbb{C}_\infty)$ is induced via (2.1) by the morphism

$$\mathbb{P}^1_{\mathbb{C}_\infty} \to \mathbb{P}^1_{\mathbb{C}_\infty}, \quad (x : y) \mapsto (dx - cy : -bx + ay)$$

of projective schemes over $\mathbb{C}_\infty$, it is analytic. By Proposition 5.24, the map $\gamma : \Omega \to \Omega$ is the restriction of this map to an admissible open subset and thus it is also analytic. $\square$

5.5. Admissibility of the covering $\{\mathcal{U}(v)\}_{v\in\mathcal{T}_0}$.

Lemma 5.28. *For any $v \in \mathcal{T}_0$, the subset $\mathcal{U}(v) \subseteq \Omega$ is an admissible open subset. Moreover, $\mathcal{U}^\star(v) \subseteq \Omega$ is an admissible affinoid open subset.*

Proof. Consider the admissible affinoid open covering $\{\Omega_{r,s}\}_{r,s\geqslant 0}$ of Ω. For any $r, s \geqslant 0$, the subset $\Omega_{r,s} \cap \mathcal{U}(v)$ is obtained by omitting finitely many distinguished closed discs from the affinoid variety $\Omega_{r,s}$. Since the centers of there discs do not lie in $\Omega_{r,s}$, each of these closed discs is defined by

$$\{x \in \Omega_{r,s} \mid |f(x)| \leqslant 1\} \quad \text{or} \quad \{x \in \Omega_{r,s} \mid |f(x)| \geqslant 1\}$$

with some $f \in \mathcal{O}(\Omega_{r,s})$. Now [BGR, Proposition 9.1.4/5] implies that $\Omega_{r,s} \cap \mathcal{U}(v)$ is an admissible open subset of $\Omega_{r,s}$. Then [BGR, §9.1.2, (G_1)] yields the lemma for $\mathcal{U}(v)$.

For $\mathcal{U}^\star(v)$, by the same reason the subset $\Omega_{r,s} \cap \mathcal{U}^\star(v)$ is a rational subdomain of $\Omega_{r,s}$, which implies that $\mathcal{U}^\star(v)$ is an admissible open subset of Ω. On the other hand, $\mathcal{U}^\star(v)$ is also obtained from a closed disc in $\mathbb{A}^1_{\mathbb{C}_\infty}$ by omitting finitely many open discs of type $D^\circ_{\mathbb{C}_\infty}(a, \rho)$ with $a \in K_\infty$. This implies that $\mathcal{U}^\star(v)$ is a rational subdomain of a closed disc, which is an affinoid variety. This concludes the proof of the lemma. $\square$

Lemma 5.29. *For any non-negative integers $r, s \geqslant 0$, the set*

$$\{v \in \mathcal{T}_0 \mid \mathcal{U}(v) \cap \Omega_{r,s} \neq \varnothing\}$$

is finite.

Proof. By Lemma 5.20, we can find a finite subset $J \subseteq K_\infty$ satisfying

$$\mathbb{P}^1(K_\infty) = D^\circ(\infty, q^{-s}) \sqcup \coprod_{a\in J} D^\circ(a, q^{-r}),$$

$$\Omega_{r,s} = \mathbb{P}^1(\mathbb{C}_\infty)\backslash \left(D^\circ_{\mathbb{C}_\infty}(\infty, q^{-s}) \sqcup \coprod_{a\in J} D^\circ_{\mathbb{C}_\infty}(a, q^{-r}) \right).$$

Write

$$D^\circ(\infty, q^{-s}) = D(\infty, q^{-s-1}) = U(e_\infty), \quad D^\circ(a, q^{-r}) = D(a, q^{-r-1}) = U(e_a)$$

with some $e_\infty, e_a \in \mathcal{T}_1^o$, and put $\Lambda = \{e_\infty, e_a\ (a \in J)\}$.

Let $v \in \mathcal{T}_0$. Suppose that there exist $e \in \Lambda$ and a half-line $H = \{w_n\}_{n\geqslant 0} \in H(e)$ satisfying $v = w_m$ for some integer $m \geqslant 2$. Put

$\tilde{e} = (w_{m-1} \to w_m)$. Since $\tilde{e} \neq e = (w_0 \to w_1)$, we have $U(\tilde{e}) \subsetneq U(e)$. Since $o(-\tilde{e}) = v$, Lemma 5.6 yields

$$\mathcal{U}(v) \subseteq \mathbb{P}^1(\mathbb{C}_\infty)\backslash\mathcal{U}(-\tilde{e}) \subseteq \mathcal{U}(e) \quad \text{and} \quad \mathcal{U}(v) \cap \Omega_{r,s} = \varnothing.$$

On the other hand, since we have

$$U(-e_\infty) = \coprod_{a \in J} U(e_a),$$

Lemma 4.26 implies that any half-line in $H(-e_\infty)$ meets e_a for some $a \in J$. Let X be the set of vertices such that these half-lines pass through up to $t(e_a)$. Then X is finite and we have $\mathcal{U}(v) \cap \Omega_{r,s} = \varnothing$ for any $v \notin X$. This concludes the proof. □

Proposition 5.30. *The coverings*

$$\{\mathcal{U}(v)\}_{v \in \mathcal{T}_0}, \quad \{\mathcal{U}^\star(v)\}_{v \in \mathcal{T}_0}$$

of Lemma 5.12 are admissible open coverings of Ω.

Proof. By (5.1) and [BGR, §9.1.2, (G_2)], it is enough to show the lemma for $\{\mathcal{U}^\star(v)\}_{v \in \mathcal{T}_0}$. By Proposition 5.24 combined with [BGR, Definition 9.1.1/1 (iii)] and [BGR, §9.1.2, (G_2)], we are reduced to showing that for any integers $r, s \geqslant 0$, the covering

$$\{\Omega_{r,s} \cap \mathcal{U}^\star(v)\}_{v \in \mathcal{T}_0} \tag{5.3}$$

is an admissible open covering of the affinoid variety $\Omega_{r,s}$.

For this, Lemma 5.29 and $\mathcal{U}^\star(v) \subseteq \mathcal{U}(v)$ imply that the covering (5.3) has a finite refinement. Since $\Omega_{r,s} \cap \mathcal{U}^\star(v)$ is obtained from $\Omega_{r,s}$ by omitting finitely many open discs, it is a rational subdomain of $\Omega_{r,s}$. Thus the covering (5.3) has a refinement which is an admissible open covering of $\Omega_{r,s}$. Hence [BGR, §9.1.2, (G_2)] implies that (5.3) is admissible. □

6. Drinfed modular forms

6.1. **Carlitz exponential.**

Definition 6.1. We say an additive subgroup $\mathfrak{a}$ of K is fractional almost-ideal if it contains a fractional ideal $\mathfrak{b}$ of A such that the index $[\mathfrak{a} : \mathfrak{b}]$ is finite [Böc, Definition 3.26]. When $\mathfrak{b}$ can be chosen to be nonzero, we say the fractional almost-ideal $\mathfrak{a}$ is proper.

Lemma 6.2. *Let* $\mathfrak{a}$ *be any proper fractional almost-ideal of* K. *Then there exists* $N_\mathfrak{a} \in \mathbb{Z}$ *such that for any integer* $r \leqslant N_\mathfrak{a}$, *we have*

$$\mathfrak{a} + \{z \in \mathbb{C}_\infty \mid |z| < q^{-r}\} = K_\infty + \{z \in \mathbb{C}_\infty \mid |z| < q^{-r}\}.$$

Proof. Replacing $\mathfrak{a}$ by a nonzero fractional ideal it contains, we may assume that $\mathfrak{a}$ is a nonzero fractional ideal of A. Take any $b \neq 0 \in A$ satisfying $b\mathfrak{a} \subseteq A$ and any $c \neq 0 \in b\mathfrak{a}$. Put $N_{\mathfrak{a}} = \deg(b) - \deg(c)$. For any integer $r \leqslant N_{\mathfrak{a}}$, we have $-r + \deg(b) - \deg(c) \geqslant 0$ and thus

$$A+\{z \in \mathbb{C}_\infty \mid |z| < q^{-r+\deg(b)-\deg(c)}\} = K_\infty+\{z \in \mathbb{C}_\infty \mid |z| < q^{-r+\deg(b)-\deg(c)}\}.$$

Multiplying c, we obtain

$$cA + \{z \in \mathbb{C}_\infty \mid |z| < q^{-r+\deg(b)}\} = K_\infty + \{z \in \mathbb{C}_\infty \mid |z| < q^{-r+\deg(b)}\}.$$

Since $cA \subseteq b\mathfrak{a}$, this yields

$$b\mathfrak{a} + \{z \in \mathbb{C}_\infty \mid |z| < q^{-r+\deg(b)}\} = K_\infty + \{z \in \mathbb{C}_\infty \mid |z| < q^{-r+\deg(b)}\}.$$

Then the lemma follows by multiplying b^{-1}. □

Definition 6.3. Let $\Lambda \subseteq \mathbb{C}_\infty$ be an additive subgroup. We say Λ is a lattice in $\mathbb{C}_\infty$ if it is discrete, namely for any $\rho \in \mathbb{R}_{>0}$ the subset $\Lambda \cap D_{\mathbb{C}_\infty}(0,\rho)$ is finite. We say an $\mathbb{F}_q$-subspace $\Lambda \subseteq \mathbb{C}_\infty$ is an $\mathbb{F}_q$-lattice if its underlying additive group is a lattice in $\mathbb{C}_\infty$.

For any lattice $\Lambda \subseteq \mathbb{C}_\infty$ and any $\rho \in \mathbb{R}_{>0}$, let

$$\Lambda^{\leqslant\rho} = \{\lambda \in \Lambda \mid |\lambda| \leqslant \rho\}, \quad \Lambda^{<\rho} = \{\lambda \in \Lambda \mid |\lambda| < \rho\}.$$

Then $\Lambda^{\leqslant\rho} \subseteq \Lambda^{\leqslant\rho'}$ and $\Lambda^{<\rho} \subseteq \Lambda^{<\rho'}$ for any $\rho \leqslant \rho'$.

Lemma 6.4. *Any fractional almost-ideal $\mathfrak{a} \subseteq K$ is a lattice in $\mathbb{C}_\infty$.*

Proof. Let $\rho \in \mathbb{R}_{>0}$. Take $b \neq 0 \in A$ satisfying $b\mathfrak{a} \subseteq A$. For any $a \in A$ satisfying $\frac{a}{b} \in \mathfrak{a}$, we have $\frac{a}{b} \in \mathfrak{a}^{\leqslant\rho}$ if and only if $\deg(a) \leqslant \log_q(\rho|b|)$, which implies that $\mathfrak{a}^{\leqslant\rho}$ is a finite set. This concludes the proof. □

For any lattice $\Lambda \subseteq \mathbb{C}_\infty$, define

$$e_{\Lambda,n}(X) := X \prod_{\lambda \in \Lambda^{\leqslant q^n}\setminus\{0\}} \left(1 - \frac{X}{\lambda}\right) \in \mathbb{C}_\infty[X].$$

Definition 6.5. For any $\rho \in q^{\mathbb{Q}}$, put

$$T_\rho = \left\{ \sum_{n\geqslant 0} a_n X^n \in \mathbb{C}_\infty[[X]] \;\middle|\; \lim_{n\to\infty} |a_n|\rho^n = 0 \right\}.$$

Then the $\mathbb{C}_\infty$-algebra T_ρ is an affinoid algebra with the ρ-Gauss norm

$$|f|_\rho = \max\{|a_n|\rho^n \mid n \geqslant 0\}.$$

By [BGR, Proposition 6.1.5/2], the ρ-Gauss norm is a valuation on T_ρ.

Lemma 6.6. *Let* $(G, |-|)$ *be a normed group* [BGR, Definition 1.1.3/1]. *Then a sequence* $\{a_n\}_{n\geqslant 0}$ *in* G *is a Cauchy sequence if and only if*

$$\lim_{n\to\infty} |a_n - a_{n+1}| = 0.$$

Proof. Suppose that $\{a_n\}_{n\geqslant 0}$ is Cauchy. Then for any $\varepsilon > 0$ there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that for any $n, m \geqslant N$ we have $|a_n - a_m| < \varepsilon$. In particular, for any $n \geqslant N$ we have $|a_n - a_{n+1}| < \varepsilon$, which means the equality of the lemma.

Conversely, suppose that we have $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{Z}_{\geqslant 0}$ such that for any $n \geqslant N$ we have $|a_n - a_{n+1}| < \varepsilon$. In particular, for any $m \geqslant n$, this yields

$$|a_n - a_m| = |(a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + \cdots + (a_{m-1} - a_m)| < \varepsilon,$$

which means that $\{a_n\}_{n\geqslant 0}$ is Cauchy. □

Lemma 6.7. *For any lattice* $\Lambda \subseteq \mathbb{C}_\infty$ *and any* $\rho \in q^{\mathbb{Q}}$*, the sequence* $\{e_{\Lambda,n}(X)\}_{n\geqslant 0}$ *converges in the affinoid algebra* T_ρ*.*

Proof. We show that $\{e_{\Lambda,n}(X)\}_{n\geqslant 0}$ is a Cauchy sequence. Fix a positive integer $d \geqslant \log_q \rho$. For any $\lambda \in \Lambda \backslash \Lambda^{\leqslant q^d}$, we have

$$\left|\frac{X}{\lambda}\right|_\rho = \frac{\rho}{|\lambda|} < 1, \quad \left|1 - \frac{z}{\lambda}\right|_\rho = 1.$$

Thus for any positive integer $n \geqslant d$ we have

$$|e_{\Lambda,n}(X)|_\rho = |e_{\Lambda,d}(X)|_\rho \prod_{a \in \Lambda^{\leqslant q^n} \backslash \Lambda^{\leqslant q^d}} \left|1 - \frac{X}{a}\right|_\rho = |e_{\Lambda,d}(X)|_\rho.$$

On the other hand, for any integers n, n' satisfying $n' \geqslant n \geqslant d$, we have

$$\begin{aligned} |e_{\Lambda,n'}(X) - e_{\Lambda,n}(X)|_\rho &= |e_{\Lambda,n}(X)|_\rho \left| \prod_{\lambda \in \Lambda^{\leqslant q^{n'}} \backslash \Lambda^{\leqslant q^n}} \left(1 - \frac{X}{\lambda}\right) - 1 \right|_\rho \\ &< |e_{\Lambda,d}(X)|_\rho q^{d-n}. \end{aligned}$$

Since $\lim_{n\to\infty} q^{d-n} = 0$, the lemma follows from Lemma 6.6. □

Definition 6.8. For any lattice $\Lambda \subseteq \mathbb{C}_\infty$ and any $\rho \in q^{\mathbb{Q}}$, consider the limit

$$\lim_{n\to\infty} e_{\Lambda,n}(X)$$

in the affinoid algebra T_ρ. Since for any $\rho < \rho'$ the natural map $T_\rho \to T_{\rho'}$ is continuous, the limit is independent of the choice of ρ and defines an element

$$\exp_\Lambda(X) \in \bigcap_{\rho \in q^{\mathbb{Q}}} T_\rho =: \mathbb{C}_\infty\{\{X\}\} = \mathcal{O}(\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_\infty}),$$

where $\mathbb{C}_\infty\{\{X\}\}$ is the ring of entire series (that is, series of infinite radius of convergence). We call $\exp_\Lambda(X)$ the Carlitz exponential for the lattice Λ.

Note that $\exp_\Lambda(X)$ has the constant term zero and the linear term X.

For any $z \in \mathbb{C}_\infty$ satisfying $|z| \leqslant \rho$, the map

$$T_\rho \to \mathbb{C}_\infty, \quad f(X) \mapsto f(z)$$

is well-defined and continuous. This implies

$$\exp_\Lambda(z) = \lim_{n\to\infty} e_{\Lambda,n}(z).$$

Lemma 6.9. *We have*

$$\exp_\Lambda(X+Y) = \exp_\Lambda(X) + \exp_\Lambda(Y).$$

Moreover, if Λ is an $\mathbb{F}_q$-lattice, then we also have

$$\exp_\Lambda(cX) = c\exp_\Lambda(X) \quad \textit{for any } c \in \mathbb{F}_q$$

and $\exp_\Lambda(X)$ can be written as

$$\exp_\Lambda(X) = X + \sum_{n>0} a_n X^{q^n}, \quad a_n \in \mathbb{C}_\infty.$$

Proof. Take any integer $n \geqslant 0$. Since $e_{\Lambda,n}(X)$ is separable and the set of roots of $e_{\Lambda,n}(X)$ in $\mathbb{C}_\infty$ is the additive subgroup $\Lambda^{\leqslant q^n}$ of $\mathbb{C}_\infty$, [Gos2, Theorem 1.2.1] implies that $e_{\Lambda,n}(X)$ is additive. Moreover, if Λ is an $\mathbb{F}_q$-lattice, then $\Lambda^{\leqslant q^n} \subseteq \mathbb{C}_\infty$ is an $\mathbb{F}_q$-subspace and by [Gos2, Corollary 1.2.2] we see that $e_{\Lambda,n}(X)$ is $\mathbb{F}_q$-linear. Then the lemma follows by taking the limit. □

Lemma 6.10. *Let $\rho \in \mathbb{Q}_{>0}$. Let $f(X) \in \mathbb{C}_\infty[[X]]$ be a rigid analytic function on $D_{\mathbb{C}_\infty}(0,\rho)$ satisfying $f(0) = 0$. Let $\sigma = |f|_{\mathrm{sup}}$, the supremum norm on $D_{\mathbb{C}_\infty}(0,\rho)$. Note that $\sigma \in \mathbb{Q}_{\geqslant 0}$ by the maximum modulus principle. Then we have*

$$f(D_{\mathbb{C}_\infty}(0,\rho)) = D_{\mathbb{C}_\infty}(0,\sigma), \quad f(D^\circ_{\mathbb{C}_\infty}(0,\rho)) = D^\circ_{\mathbb{C}_\infty}(0,\sigma).$$

Proof. By composing f with the isomorphism of rigid analytic varieties

$$D_{\mathbb{C}_\infty}(0,1) \simeq D_{\mathbb{C}_\infty}(0,\rho), \quad x \mapsto \varpi x$$

with $\varpi \in \mathbb{C}_\infty$ satisfying $|\varpi| = \rho$, we may assume $\rho = 1$.

Let us show the first equality. Since $|f(x)| \leqslant \sigma$ for any $x \in D_{\mathbb{C}_\infty}(0,1)$, we have $f(D_{\mathbb{C}_\infty}(0,1)) \subseteq D_{\mathbb{C}_\infty}(0,\sigma)$. Conversely, take any $y \in D_{\mathbb{C}_\infty}(0,\sigma)$. Write

$$f(X) = \sum_{n \geqslant 1} a_n X^n, \quad a_n \in \mathbb{C}_\infty$$

and consider its Newton polygon. By [BGR, Corollary 5.1.4/6], we have $\sigma = \max\{|a_n| \mid n \geqslant 1\}$. Thus the Newton polygon of $f(X) - y$ has at least one segment of slope $\leqslant 0$, which corresponds to an element $x \in D_{\mathbb{C}_\infty}(0,1)$ satisfying $f(x) = y$. This yields the first equality.

For the second equality, take any $x \in D^\circ_{\mathbb{C}_\infty}(0,1)$. Let b be the y-intercept of the tangent line of the Newton polygon of slope $-v_\infty(x)$. Then we have $v_\infty(f(x)) \geqslant b$, which implies $f(D^\circ_{\mathbb{C}_\infty}(0,1)) \subseteq D^\circ_{\mathbb{C}_\infty}(0,\sigma)$. By inspecting the Newton polygon, it also follows that if $|y| < \sigma$, then any $x \in D_{\mathbb{C}_\infty}(0,1)$ with $f(x) = y$ satisfies $|x| < 1$. This concludes the proof. □

Definition 6.11. For any lattice $\Lambda \subseteq \mathbb{C}_\infty$ and any $\rho \in \mathbb{R}_{>0}$, we put

$$\sigma_{\Lambda,\rho} := \rho \prod_{0 \neq a \in \Lambda^{<\rho}} \frac{\rho}{|a|}.$$

Then we have $\sigma_{\Lambda,\rho} \geqslant \rho$.

Lemma 6.12. *For any lattice $\Lambda \subseteq \mathbb{C}_\infty$ and any $\rho \in q^{\mathbb{Q}}$, we have*

$$\sigma_{\Lambda,\rho} = \sup\{|\exp_\Lambda(z)| \mid z \in D_{\mathbb{C}_\infty}(0,\rho)\}.$$

In particular, if $\rho \geqslant \rho'$ then $\sigma_{\Lambda,\rho} \geqslant \sigma_{\Lambda,\rho'}$.

Proof. By [BGR, Proposition 6.1.5/5], the supremum norm of $\exp_\Lambda(X)$ on $D_{\mathbb{C}_\infty}(0,\rho)$ coincides with $|\exp_\Lambda(X)|_\rho$, which is equal to $\lim_{n\to\infty} |e_{\Lambda,n}(X)|_\rho$ by continuity.

Take any positive integer $n \geqslant \log_q \rho$. Note that $|1 - \frac{X}{\lambda}|_\rho = 1$ for any $\lambda \in \Lambda \backslash \Lambda^{\leqslant\rho}$. Since the ρ-Gauss norm is a valuation, we have

$$\begin{aligned} |e_{\Lambda,n}(X)|_\rho &= \rho \prod_{0 \neq \lambda \in \Lambda^{\leqslant\rho}} \left|1 - \frac{X}{\lambda}\right|_\rho \\ &= \rho \prod_{0 \neq \lambda \in \Lambda^{<\rho}} \left|1 - \frac{X}{\lambda}\right|_\rho \prod_{\lambda \in \Lambda,\ |\lambda| = \rho} \left|1 - \frac{X}{\lambda}\right|_\rho \\ &= \rho \prod_{0 \neq \lambda \in \Lambda^{<\rho}} \frac{\rho}{|\lambda|} = \sigma_{\Lambda,\rho}. \end{aligned}$$

By taking the limit, the equality of the lemma follows. If $\rho \geqslant \rho'$, then we have $D_{\mathbb{C}_\infty}(0,\rho) \supseteq D_{\mathbb{C}_\infty}(0,\rho')$, which yields the last assertion. □

Corollary 6.13. *For any lattice $\Lambda \subseteq \mathbb{C}_\infty$ and any $\rho \in q^{\mathbb{Q}}$, we have*

$$\exp_\Lambda(D_{\mathbb{C}_\infty}(0,\rho)) = D_{\mathbb{C}_\infty}(0,\sigma_{\Lambda,\rho}), \quad \exp_\Lambda(D^\circ_{\mathbb{C}_\infty}(0,\rho)) = D^\circ_{\mathbb{C}_\infty}(0,\sigma_{\Lambda,\rho}).$$

Proof. This follows from Lemma 6.10 and Lemma 6.12. □

Lemma 6.14. *For any lattice $\Lambda \subseteq \mathbb{C}_\infty$, the sequence of additive groups*

$$0 \longrightarrow \Lambda \longrightarrow \mathbb{C}_\infty \xrightarrow{\exp_\Lambda} \mathbb{C}_\infty \longrightarrow 0$$

is exact.

Proof. First we show $\mathrm{Ker}(\exp_\Lambda) = \Lambda$. For any $\lambda \in \Lambda$, we have $e_{\Lambda,n}(\lambda) = 0$ for all $n \geqslant \log_q(|\lambda|)$ and thus $\exp_\Lambda(\lambda) = 0$. Conversely, for any $z \in \mathbb{C}_\infty \backslash \Lambda$ we have $e_{\Lambda,n}(z) \neq 0$ for all n. As in the proof of Lemma 6.7, we can show that the absolute value $|e_{\Lambda,n}(z)|$ is stable for any n satisfying $|z| < q^n$. This yields $\exp_\Lambda(z) \neq 0$.

On the other hand, since $\sigma_{\Lambda,\rho} \geqslant \rho$ we have $\lim_{\rho\to\infty} \sigma_{\Lambda,\rho} = \infty$. Thus Corollary 6.13 implies that $\exp_\Lambda$ is surjective. This concludes the proof. □

Definition 6.15. Let $\mathbb{G}_\mathrm{a} = \mathrm{Spec}(\mathbb{C}_\infty[X])$ be the additive group. We denote by C the Drinfeld module of rank one defined by the homomorphism of $\mathbb{F}_q$-algebras

$$\Phi^C : A \to \mathrm{End}(\mathbb{G}_\mathrm{a}), \quad t \mapsto (X \mapsto tX + X^q)$$

and call it the Carlitz module. For any $a \in A$, we write $\Phi^C_a(X) := \Phi^C(a)^*(X)$.

Definition 6.16. We fix once and for all a $(q-1)$-st root of $-t$ in $\mathbb{C}_\infty$ and put

$$\bar{\pi} := (-t)^{\frac{q}{q-1}} \prod_{n\geqslant 1} (1 - t^{1-q^n})^{-1}.$$

We call $\bar{\pi}$ the Carlitz period.

Note that $\bar{\pi}A$ is an $\mathbb{F}_q$-lattice in $\mathbb{C}_\infty$. We define

$$e_C(X) := \exp_{\bar{\pi}A}(X) = \bar{\pi}\exp_A(\bar{\pi}^{-1}X).$$

Proposition 6.17. *For any $a \in A$, we have*

$$\bar{\pi}\exp_A(aX) = \Phi^C_a(\bar{\pi}\exp_A(X))$$

in the ring $\mathbb{C}_\infty\{\{X\}\}$.

Proof. By [Gos2, Proposition 3.3.1], we have

$$e_C(aX) = \Phi^C_a(e_C(X)),$$

from which we obtain

$$\bar{\pi}\exp_A(aX) = e_C(a\bar{\pi}X) = \Phi^C_a(e_C(\bar{\pi}X)) = \Phi^C_a(\bar{\pi}\exp_A(X)).$$

□

6.2. **Quotient by a discrete group action.** Let $\mathbb{K}$ be a field equipped with a complete non-Archimedean valuation $|-|:\mathbb{K}\to\mathbb{R}_{\geqslant 0}$. Let B be an affinoid algebra over $\mathbb{K}$ and let G be a finite group which acts on the $\mathbb{K}$-affinoid variety $\mathrm{Sp}(B)$ from the left. Write the induced right action of $g\in G$ on B as $b\mapsto b|_g$. We denote by

$$B^G := \{b\in B \mid b|_g = b \text{ for any } g\in G\}$$

the subring of G-invariants in B.

Lemma 6.18. *(1) The ring B^G is an affinoid algebra over $\mathbb{K}$ and the map $B^G\to B$ is finite.*

(2) Let $\pi:\mathrm{Sp}(B)\to\mathrm{Sp}(B^G)$ be the natural morphism of affinoid varieties. Then it is a G-invariant surjection such that for any $x\in\mathrm{Sp}(B)$ the fiber $\pi^{-1}(\pi(x))$ agrees with the G-orbit of x.

(3) The map π induces a bijection

$$G\backslash\mathrm{Sp}(B)\to\mathrm{Sp}(B^G).$$

(4) For any $x\in\mathrm{Sp}(B)$ and $y\in\mathrm{Sp}(B^G)$, we denote by $B^\wedge_x$ and $(B^G)^\wedge_y$ their complete local rings at x and y, respectively. Then the natural map

$$(B^G)^\wedge_y\to\Big(\prod_{\pi(x)=y}B^\wedge_x\Big)^G$$

is an isomorphism of $\mathbb{K}$-algebras.

(5) For any affinoid subdomain $\mathrm{Sp}(C)\subseteq\mathrm{Sp}(B^G)$, we have a natural isomorphism $C\to(B\otimes_{B^G}C)^G$. In particular, the natural morphism of sheaves $\mathcal{O}_{\mathrm{Sp}(B^G)}\to(\pi_*\mathcal{O}_{\mathrm{Sp}(B)})^G$ is an isomorphism.

Proof. The assertion (1) follows from [BGR, Proposition 6.3.3/3]. Since the map $B^G\to B$ is a finite injection, we see that π is surjective. Then [Sta, Lemma 15.110.8] yields (2), which implies (3).

Consider the exact sequence of B^G-modules

$$0\longrightarrow B^G\longrightarrow B\longrightarrow \textstyle\prod_{g\in G}B, \tag{6.1}$$

where the last map is given by $b\mapsto(b|_g-b)_{g\in G}$. For any flat B^G-algebra C, this induces the exact sequence

$$0\longrightarrow C\longrightarrow B\otimes_{B_G}C\longrightarrow \textstyle\prod_{g\in G}(B\otimes_{B_G}C)\,.$$

Hence we see that the natural map

$$C\to(B\otimes_{B_G}C)^G$$

is an isomorphism. By [BGR, Proposition 7.2.2/1], this yields (5).

Moreover, for any $y \in \mathrm{Sp}(B^G)$, applying this to $C = (B^G)^\wedge_y$ we obtain an isomorphism

$$(B^G)^\wedge_y \to \left(B \otimes_{B_G} (B^G)^\wedge_y\right)^G.$$

Since B^G-algebra B is finite, Hensel's lemma gives a natural isomorphism of $(B^G)^\wedge_y$-algebras

$$B \otimes_{B_G} (B^G)^\wedge_y \simeq \prod_{\pi(x)=y} B^\wedge_x,$$

from which (4) follows. This concludes the proof. □

Lemma 6.19 ([Dri], Proposition 6.4)**.** *Let* $\mathrm{Sp}(C) \subseteq \mathrm{Sp}(B)$ *be an affinoid subdomain which is stable under the action of* G*. Then, via the natural map* $\mathrm{Sp}(C^G) \to \mathrm{Sp}(B^G)$*, the affinoid variety* $\mathrm{Sp}(C^G)$ *is an affinoid subdomain of* $\mathrm{Sp}(B^G)$*.*

Proof. Consider the natural commutative diagram of affinoid varieties

$$\begin{tikzcd} \mathrm{Sp}(C) \arrow[r, "j"] \arrow[d, "\pi_C"'] & \mathrm{Sp}(B) \arrow[d, "\pi_B"] \\ \mathrm{Sp}(C^G) \arrow[r, "\bar{j}"'] & \mathrm{Sp}(B^G), \end{tikzcd}$$

where j is an open immersion. By [BGR, Corollary 8.2.1/4], it is enough to show that $\bar{j}$ is an open immersion.

By Lemma 6.18 (3), on the underlying sets the map $\bar{j}$ can be identified with the natural map $G\backslash\mathrm{Sp}(C) \to G\backslash\mathrm{Sp}(B)$. Since $\mathrm{Sp}(C) \subseteq \mathrm{Sp}(B)$ is G-stable, it follows that $\bar{j}$ is injective.

On the other hand, let $y \in \mathrm{Sp}(C^G)$ and $y' = \bar{j}(y)$. By Lemma 6.18 (2), the injection j induces a bijection $\pi_C^{-1}(y) \to \pi_B^{-1}(y')$. By [BGR, Proposition 7.2.2/1 (ii)], this gives a G-equivariant isomorphism of $\mathbb{K}$-algebras

$$j^* : \prod_{\pi_B(x')=y'} B^\wedge_{x'} \to \prod_{\pi_C(x)=y} C^\wedge_x.$$

Then Lemma 6.18 (4) shows that the natural map $(B^G)^\wedge_{y'} \to (C^G)^\wedge_y$ is an isomorphism. Now [BGR, Proposition 7.3.3/5] concludes the proof. □

Lemma 6.20 ([Dri], §6, B)**.** *Let* Z *be a separated rigid analytic variety over* $\mathbb{K}$ *and let* $\phi : \mathrm{Sp}(B) \to Z$ *be a morphism of rigid analytic varieties over* $\mathbb{K}$ *which is* G*-invariant. Then there exists a unique morphism* $\psi : \mathrm{Sp}(B^G) \to Z$ *satisfying* $\phi = \psi \circ \pi$*.*

Proof. Write $X = \mathrm{Sp}(B)$ and $Y = \mathrm{Sp}(B^G)$. By Lemma 6.18, the morphism $\pi : X \to Y$ is defined by a pair of a surjection between underlying sets and an injection $\mathcal{O}_Y \to \pi_* \mathcal{O}_X$. This gives the uniqueness of ψ.

Let us show the existence of ψ. Let $Z = \bigcup_{i \in I} Z_i$ be an admissible affinoid open covering, so that $X_i := \phi^{-1}(Z_i)$ is a G-stable admissible open subset of X and $\{X_i\}_{i\in I}$ forms an admissible open covering of X. By [BGR, Proposition 9.1.4/2 (ii)], the latter covering has a finite subcovering.

Since Z is separated, the diagonal map $\Delta_Z : Z \to Z \times Z$ is a closed immersion and the cartesian diagram

$$\begin{tikzcd} X_i \arrow[r, hook] \arrow[d] & X \arrow[r, "\phi"] \arrow[d, "(\mathrm{id},\phi)"'] & Z \arrow[d, "\Delta_Z"] \\ X \times Z_i \arrow[r, hook] & X \times Z \arrow[r, "\phi\times\mathrm{id}"'] & Z \times Z \end{tikzcd}$$

implies that X_i is an affinoid variety on which G acts from the left.

Write $X_i = \mathrm{Sp}(B_i)$. Consider the natural finite surjection

$$\pi_i : X_i = \mathrm{Sp}(B_i) \to \mathrm{Sp}(B_i^G) =: Y_i.$$

Since X_i is an affinoid subdomain of X, Lemma 6.19 implies that Y_i is an affinoid subdomain of Y and we have a commutative diagram

$$\begin{tikzcd} X_i = \mathrm{Sp}(B_i) \arrow[r, "\pi_i"] \arrow[d, hook] & Y_i = \mathrm{Sp}(B_i^G) \arrow[d, hook] \\ X = \mathrm{Sp}(B) \arrow[r, "\pi"'] & Y = \mathrm{Sp}(B^G). \end{tikzcd} \tag{6.2}$$

Since X_i is G-stable, Lemma 6.18 (2) gives $X_i = \pi^{-1}(Y_i)$. Since the covering $\{X_i\}_{i\in I}$ of X has a finite subcovering, we see that $\{Y_i\}_{i\in I}$ covers Y and has a finite subcovering. In particular, the latter covering is an admissible open covering of Y.

Since ϕ is G-invariant, the map $\phi^* : \mathcal{O}(Z_i) \to B_i$ factors through the subring B_i^G and thus we have a morphism $\psi_i : Y_i \to Z_i$ satisfying $\phi|_{X_i} = \psi_i \circ \pi_i$.

Now we claim $\psi_i|_{Y_i \cap Y_j} = \psi_j|_{Y_i \cap Y_j}$ for any $i, j \in I$. Note that $Y_i \cap Y_j$ is an affinoid subdomain of Y. By Lemma 6.18 (5), the morphism $\pi^{-1}(Y_i \cap Y_j) \to Y_i \cap Y_j$ is identified with the quotient of the affinoid variety $\pi^{-1}(Y_i \cap Y_j)$ by G. Since the map on their affinoid algebras is injective, it is an epimorphism in the category of $\mathbb{K}$-affinoid varieties.

Moreover, the commutative diagram (6.2) and $X_i = \pi^{-1}(Y_i)$ yield

$$\pi_i^{-1}(Y_i \cap Y_j) = \pi^{-1}(Y_i \cap Y_j) = \pi_j^{-1}(Y_i \cap Y_j).$$

Thus we obtain

$$\psi_i|_{Y_i \cap Y_j} \circ \pi|_{\pi^{-1}(Y_i \cap Y_j)} = \phi|_{\pi^{-1}(Y_i \cap Y_j)} = \psi_j|_{Y_i \cap Y_j} \circ \pi|_{\pi^{-1}(Y_i \cap Y_j)}.$$

Since $\pi|_{\pi^{-1}(Y_i \cap Y_j)} : \pi^{-1}(Y_i \cap Y_j) \to Y_i \cap Y_j$ is an epimorphism of $\mathbb{K}$-affinoid varieties, the claim follows.

Therefore, by [BGR, Proposition 9.3.3/1] we can glue the morphisms ψ_i to obtain a morphism of rigid analytic varieties $\psi : Y \to Z$ satisfying $\phi = \psi \circ \pi$. This concludes the proof of the lemma. $\square$

Definition 6.21. Let X be a separated rigid analytic variety over $\mathbb{K}$. Let Γ be a group which acts on X from the left. We say that the action of Γ on X is discrete if there exist an admissible affinoid open covering $X = \bigcup_{i \in I} X_i$ and an action of Γ on I satisfying the following conditions.

(1) For any $\gamma \in \Gamma$ and $i \in I$, we have $\gamma(X_i) = X_{\gamma(i)}$.
(2) For any $i \in I$, the subgroup

$$\Gamma_i = \{\gamma \in \Gamma \mid \gamma(X_i) = X_i\} = \mathrm{Stab}_\Gamma(i)$$

is finite.
(3) (a) If $\gamma \notin \Gamma_i$, then $X_i \cap X_{\gamma(i)} = \varnothing$.
(b) For any $i, j \in I$, the subset

$$\Gamma_{i,j} = \{\gamma \in \Gamma \mid X_i \cap \gamma(X_j) \neq \varnothing\}$$

is finite.
(4) For any $i \in I$, the subset

$$\Gamma X_i := \bigcup_{\gamma \in \Gamma} \gamma(X_i)$$

is an admissible open subset of X, and the covering $\{\gamma(X_i)\}_{\gamma \in \Gamma}$ is its admissible open covering.

Let X, Γ and $\{X_i\}_{i \in I}$ satisfy the conditions of Definition 6.21. Then for any $i, j \in I$, we have

$$\Gamma_{j,i} = \{\gamma^{-1} \mid \gamma \in \Gamma_{i,j}\}. \tag{6.3}$$

Moreover, the group Γ_i acts on $\Gamma_{i,j}$ from the left, and the group Γ_j acts on $\Gamma_{i,j}$ from the right.

Let $\pi : X \to Y := \Gamma \backslash X$ be the quotient map. We will give Y a structure of a rigid analytic variety over $\mathbb{K}$.

Lemma 6.22. *For any $i, j \in I$, the natural map $\pi : X \to Y$ induces bijections*

$$\Gamma_i \backslash X_i \to \pi(X_i), \quad \Gamma_i \cap \Gamma_j \backslash X_i \cap X_j \to \pi(X_i \cap X_j).$$

Proof. Let $x, x' \in X_i$. If $\gamma \in \Gamma$ satisfies $\gamma(x) = x'$, then $x' \in X_i \cap X_{\gamma(i)}$ and thus $\gamma \in \Gamma_i$. This concludes the proof. □

Write $X_i = \mathrm{Sp}(B_i)$. Define a structure of an affinoid variety over $\mathbb{K}$ on $Y_i := \pi(X_i)$ via the bijection

$$\rho_i : \mathrm{Sp}(B_i^{\Gamma_i}) \simeq \Gamma_i \backslash \mathrm{Sp}(B_i) = \Gamma_i \backslash X_i \xrightarrow{\pi} \pi(X_i) = Y_i,$$

where the first arrow is the natural bijection of Lemma 6.18 (3). Then the natural map $X_i \to \pi(X_i)$ is identified with the underlying map of the affinoid morphism associated with the natural inclusion $B_i^{\Gamma_i} \to B_i$. Note that the rigid analytic structure on $\pi(X_i)$ depends on the choice of $i \in I$.

On the other hand, since X is assumed to be separated, for any $i, j, k \in I$ we see that $X_i \cap X_j$ and $X_i \cap X_j \cap X_k$ are affinoid subdomains of X_i. Write

$$X_i \cap X_j = \mathrm{Sp}(B_{i,j}), \quad X_i \cap X_j \cap X_k = \mathrm{Sp}(B_{i,j,k}),$$

so that $B_{i,j} = B_{j,i}$. Note that the affinoid subdomain $X_i \cap X_j \subseteq X_i$ is stable under the action of $\Gamma_i \cap \Gamma_j$. Then we have a commutative diagram of affinoid varieties

(6.4)
$$\begin{tikzcd} \mathrm{Sp}(B_{i,j}) \arrow[r] \arrow[d] & \mathrm{Sp}(B_{i,j}^{\Gamma_i \cap \Gamma_j}) \arrow[d, "\iota_{i,j}"] \\ \mathrm{Sp}(B_i) \arrow[r] & \mathrm{Sp}(B_i^{\Gamma_i}). \end{tikzcd}$$

Lemma 6.23. *The map $\iota_{i,j} : \mathrm{Sp}(B_{i,j}^{\Gamma_i \cap \Gamma_j}) \to \mathrm{Sp}(B_i^{\Gamma_i})$ is an open immersion.*

Proof. By Lemma 6.22, the map $\iota_{i,j}$ is injective. Take any $y \in \mathrm{Sp}(B_{i,j}^{\Gamma_i \cap \Gamma_j})$ and write $z = \iota_{i,j}(y)$. Choose $x \in X_i \cap X_j \subseteq X_i$ lying over y. By Lemma 6.18 (4), we have natural isomorphisms between complete local rings

$$(B_i^{\Gamma_i})_z^\wedge \to \left(\prod_{w \in \Gamma_i x} B_{i,w}^\wedge \right)^{\Gamma_i}, \quad (B_{i,j}^{\Gamma_i \cap \Gamma_j})_y^\wedge \to \left(\prod_{w \in (\Gamma_i \cap \Gamma_j) x} B_{i,j,w}^\wedge \right)^{\Gamma_i \cap \Gamma_j}.$$

By [BGR, Proposition 7.3.3/5], it is enough to show that the natural map $(B_i^{\Gamma_i})_z^\wedge \to (B_{i,j}^{\Gamma_i \cap \Gamma_j})_y^\wedge$ is an isomorphism.

Put $\Gamma_i(x) := \mathrm{Stab}_{\Gamma_i}(x)$ and $(\Gamma_i \cap \Gamma_j)(x) := \mathrm{Stab}_{\Gamma_i\cap\Gamma_j}(x)$. Then we have a commutative diagram of complete local rings

$$\begin{tikzcd}
(B^{\wedge}_{i,x})^{\Gamma_i(x)} \arrow[r] \arrow[d] & \left(\prod_{w\in\Gamma_i x} B^{\wedge}_{i,w}\right)^{\Gamma_i} \arrow[d] \\
(B^{\wedge}_{i,j,x})^{(\Gamma_i\cap\Gamma_j)(x)} \arrow[r] & \left(\prod_{w\in(\Gamma_i\cap\Gamma_j)x} B^{\wedge}_{i,j,w}\right)^{\Gamma_i\cap\Gamma_j},
\end{tikzcd}$$

where horizontal arrows are isomorphisms. Since $\mathrm{Sp}(B_{i,j})$ is an affinoid subdomain of $\mathrm{Sp}(B_i)$, the natural map $B^{\wedge}_{i,x} \to B^{\wedge}_{i,j,x}$ is an isomorphism. Thus we are reduced to showing $\Gamma_i(x) = (\Gamma_i \cap \Gamma_j)(x)$.

Since $(\Gamma_i \cap \Gamma_j)(x) \subseteq \Gamma_i(x)$, it is enough to show the reverse containment. Let $\gamma \in \Gamma_i$ satisfy $\gamma(x) = x$. Since $x \in X_j$ and $x = \gamma(x) \in X_j \cap X_{\gamma(j)}$, Definition 6.21 (3) yields $\gamma \in \Gamma_j$ and $\gamma \in (\Gamma_i \cap \Gamma_j)(x)$. This concludes the proof. □

Consider the natural bijection

$$\rho_{i,j} : \mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}) \simeq \Gamma_i \cap \Gamma_j\backslash \mathrm{Sp}(B_{i,j}) = \Gamma_i \cap \Gamma_j\backslash X_i \cap X_j \xrightarrow{\pi} \pi(X_i \cap X_j),$$

so that $\rho_{i,j} = \rho_{j,i}$. We have a commutative diagram of sets

$$\begin{tikzcd}
\mathrm{Sp}(B_{i,j}) \arrow[r] \arrow[d] & \mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}) \arrow[r, "\rho_{i,j}", "\sim"'] \arrow[d, "\iota_{i,j}"] & \pi(X_i \cap X_j) \arrow[d] \\
\mathrm{Sp}(B_i) \arrow[r] & \mathrm{Sp}(B_i^{\Gamma_i}) \arrow[r, "\rho_i", "\sim"'] & \pi(X_i),
\end{tikzcd} \tag{6.5}$$

where the composites of horizontal arrows are π and the right vertical arrow is the natural inclusion. Then Lemma 6.23 and [BGR, Corollary 8.2.1/4] imply that $\pi(X_i \cap X_j)$ is an affinoid subdomain of $\pi(X_i)$ via the affinoid map $\iota_{i,j}$.

For any $i \in I$, define an equivalence relation $\sim_i$ on I by

$$j \sim_i k \quad \Leftrightarrow \quad \pi(X_i \cap X_j) = \pi(X_i \cap X_k).$$

Though the rigid analytic structure on $\pi(X_i \cap X_j)$ depends on the choice of a representative of the class of j, the universality of affinoid subdomains implies that it is unique up to a unique isomorphism. Namely, for any $j, k \in I$ satisfying $j \sim_i k$, we have a unique isomorphism $\eta_{i,j,k} : \mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}) \to \mathrm{Sp}(B_{i,k}^{\Gamma_i\cap\Gamma_k})$ which makes the following diagram commutative.

$$\begin{tikzcd} \mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}) \arrow[r, "\iota_{i,j}"] \arrow[d, "\eta_{i,j,k}"', "\wr"] & \mathrm{Sp}(B_i^{\Gamma_i}) \\ \mathrm{Sp}(B_{i,k}^{\Gamma_i\cap\Gamma_k}) \arrow[ur, "\iota_{i,k}"'] & \end{tikzcd}$$

By the uniqueness, for any j, k, l in the same class, we have

$$\eta_{i,j,k} = \eta_{i,l,k} \circ \eta_{i,j,l}. \tag{6.6}$$

Thus we may fix one of these mutually compatible rigid analytic structures. For this, fix a complete set of representatives $I(i)$ of the quotient set $I/\sim_i$. For any $j \in I$, let $j_i \in I(i)$ be the unique element satisfying $j \sim_i j_i$. We consider $\pi(X_i \cap X_j)$ as an affinoid variety via ι_{i,j_i}, so that the affinoid ring of $\pi(X_i \cap X_j)$ is $B_{i,j_i}^{\Gamma_i\cap\Gamma_{j_i}}$. Then it follows that the map $\rho_{i,j}$ can be extended to an isomorphism of affinoid varieties satisfying the commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(B_{i,j_i}^{\Gamma_i\cap\Gamma_{j_i}}) \arrow[r, "\rho_{i,j_i}", "\sim"'] \arrow[d, "\eta_{i,j_i,j}"', "\wr"] & \pi(X_i \cap X_j) \\ \mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}). \arrow[ur, "\rho_{i,j}"'] & \end{tikzcd}$$

By (6.6), this shows that for any $j, k \in I$ satisfying $j \sim_i k$, we also have the commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}) \arrow[r, "\rho_{i,j}"] \arrow[d, "\eta_{i,j,k}"', "\wr"] & \pi(X_i \cap X_j) \\ \mathrm{Sp}(B_{i,k}^{\Gamma_i\cap\Gamma_k}). \arrow[ur, "\rho_{i,k}"'] & \end{tikzcd} \tag{6.7}$$

Lemma 6.24. *For any $i, j, k \in I$, the subset $\pi(X_i \cap X_j \cap X_k)$ is an affinoid subdomain of $\pi(X_i)$ such that all affinoid maps into $\pi(X_i \cap X_j \cap X_k)$ are represented by an open immersion* $\mathrm{Sp}(B_{i,j,k}^{\Gamma_i\cap\Gamma_j\cap\Gamma_k}) \to \mathrm{Sp}(B_i^{\Gamma_i})$.

Proof. This follows similarly to Lemma 6.23. □

As before, by choosing one of the mutually compatible rigid analytic structures on $\pi(X_i \cap X_j \cap X_k)$ as an affinoid subdomain of $\pi(X_i)$, we obtain a commutative diagram of affinoid varieties

$$
\begin{tikzcd}
\mathrm{Sp}(B_{i,j,k}^{\Gamma_i\cap\Gamma_j\cap\Gamma_k}) \arrow[r, "\rho_{i,j,k}", "\sim"'] \arrow[d] & \pi(X_i\cap X_j\cap X_k) \arrow[d] \\
\mathrm{Sp}(B_{i,j}^{\Gamma_i\cap\Gamma_j}) \arrow[r, "\sim", "\rho_{i,j}"'] & \pi(X_i\cap X_j),
\end{tikzcd}
\tag{6.8}
$$

where the left vertical arrow is induced by the natural map $B_{i,j}^{\Gamma_i\cap\Gamma_j}\to B_{i,j,k}^{\Gamma_i\cap\Gamma_j\cap\Gamma_k}$ and the underlying map of the right vertical arrow is the natural inclusion.

Lemma 6.25. *For any $i,j\in I$, we have*

$$Y_{i,j}:=\pi(X_i)\cap\pi(X_j)=\coprod_{\gamma\in\Gamma_i\backslash\Gamma_{i,j}/\Gamma_j}\pi(X_i\cap X_{\gamma(j)}).$$

In particular, $Y_{i,j}$ is an affinoid subdomain of $Y_i=\pi(X_i)$.

Proof. First note that

$$\pi(X_i)\cap\pi(X_j)=\pi(X_i\cap\Gamma X_j)=\bigcup_{\gamma\in\Gamma}\pi(X_i\cap X_{\gamma(j)})=\bigcup_{\gamma\in\Gamma_{i,j}}\pi(X_i\cap X_{\gamma(j)}).$$

Moreover, for any $\gamma\in\Gamma_{i,j}$, the set $\pi(X_i\cap X_{\gamma(j)})$ depends only on the image of γ in $\Gamma_i\backslash\Gamma_{i,j}/\Gamma_j$.

Let $\gamma,\delta\in\Gamma_{i,j}$. If $\pi(X_i\cap X_{\gamma(j)})\cap\pi(X_i\cap X_{\delta(j)})\neq\emptyset$, then we can find $x\in X_i\cap X_{\gamma(j)}$, $x'\in X_i\cap X_{\delta(j)}$ and $\mu\in\Gamma$ satisfying $\mu(x)=x'$. Then $x'\in X_i\cap X_{\delta(j)}\cap X_{\mu(i)}\cap X_{\mu\gamma(j)}$, so that $\mu\in\Gamma_i$ and $\mu\gamma\in\delta\Gamma_j$. This yields $\gamma\in\Gamma_i\delta\Gamma_j$ and thus $\pi(X_i\cap X_{\gamma(j)})=\pi(X_i\cap X_{\delta(j)})$. Hence we obtain the claimed decomposition. The last assertion follows from [BGR, Proposition 7.2.2/9]. □

For any $i,j\in I$ and $\gamma\in\Gamma_{i,j}$, consider the isomorphism of affinoid varieties

$$\gamma^{-1}:X_i\cap X_{\gamma(j)}\to X_j\cap X_{\gamma^{-1}(i)}.$$

Since $\Gamma_{\gamma^{-1}(i)}=\gamma^{-1}\Gamma_i\gamma$ and $\Gamma_{\gamma(j)}=\gamma\Gamma_j\gamma^{-1}$, this induces an isomorphism of affinoid varieties

$$\gamma^{-1}:\mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i\cap\Gamma_{\gamma(j)}})\to\mathrm{Sp}(B_{j,\gamma^{-1}(i)}^{\Gamma_j\cap\Gamma_{\gamma^{-1}(i)}}).$$

Hence, there exists a unique isomorphism of affinoid varieties

$$\theta_{i,j,\gamma}:\pi(X_i\cap X_{\gamma(j)})\to\pi(X_j\cap X_{\gamma^{-1}(i)})$$

that makes the following diagram commutative:

$$\begin{tikzcd}
\pi(X_i \cap X_{\gamma(j)}) \arrow[r, "\theta_{i,j,\gamma}"] & \pi(X_j \cap X_{\gamma^{-1}(i)}) \\
\mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}}) \arrow[u, "\rho_{i,\gamma(j)}", "\wr"'] \arrow[r, "\gamma^{-1}"'] & \mathrm{Sp}(B_{j,\gamma^{-1}(i)}^{\Gamma_j \cap \Gamma_{\gamma^{-1}(i)}}). \arrow[u, "\wr", "\rho_{j,\gamma^{-1}(i)}"']
\end{tikzcd}$$

Then the uniqueness yields

$$\theta_{i,j,\gamma}^{-1} = \theta_{j,i,\gamma^{-1}}. \tag{6.9}$$

Since the composite of $\rho_{i,\gamma(j)}$ with the natural map $\mathrm{Sp}(B_{i,\gamma(j)}) \to \mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}})$ is π, it follows that $\theta_{i,j,\gamma} = \mathrm{id}$ on the level of underlying sets.

Lemma 6.26. *For any $i, j \in I$ and $\gamma \in \Gamma$, the following diagram is commutative.*

$$\begin{tikzcd}
\mathrm{Sp}(B_i^{\Gamma_i}) \arrow[r, "\gamma^{-1}"] & \mathrm{Sp}(B_{\gamma^{-1}(i)}^{\Gamma_{\gamma^{-1}(i)}}) \\
\mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}}) \arrow[u, "\iota_{i,\gamma(j)}"] \arrow[r, "\gamma^{-1}"'] & \mathrm{Sp}(B_{\gamma^{-1}(i),j}^{\Gamma_{\gamma^{-1}(i)} \cap \Gamma_j}) \arrow[u, "\iota_{\gamma^{-1}(i),j}"']
\end{tikzcd}$$

In particular, for any $\gamma \in \Gamma_i$ we have $j \sim_i \gamma(j)$ and $\eta_{i,\gamma(j),j} = \gamma^{-1}$.

Proof. Since the natural map $\mathrm{Sp}(B_{i,\gamma(j)}) \to \mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}})$ is an epimorphism in the category of affinoid varieties, by (6.4) the lemma follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}(B_i) \arrow[r, "\gamma^{-1}"] & \mathrm{Sp}(B_{\gamma^{-1}(i)}) \\
\mathrm{Sp}(B_{i,\gamma(j)}) \arrow[u, hook] \arrow[r, "\gamma^{-1}"'] & \mathrm{Sp}(B_{\gamma^{-1}(i),j}). \arrow[u, hook]
\end{tikzcd}$$

□

Lemma 6.27. *For any $i, j \in I$, the morphism $\theta_{i,j,\gamma}$ depends only on the class of γ in the coset space $\Gamma_i \backslash \Gamma_{i,j} / \Gamma_j$.*

Proof. Take any $\gamma \in \Gamma_{i,j}$, $\mu \in \Gamma_i$ and $\nu \in \Gamma_j$. Put $\delta = \mu\gamma\nu$. By Lemma 6.26 and (6.7), we have the commutative diagram of affinoid varieties

$$\begin{tikzcd}
\pi(X_i \cap X_{\delta(j)}) & \pi(X_j \cap X_{\delta^{-1}(i)}) \\
\mathrm{Sp}(B_{i,\delta(j)}^{\Gamma_i \cap \Gamma_{\delta(j)}}) \arrow[u, "\rho_{i,\delta(j)}"] \arrow[r, "\delta^{-1}"] \arrow[d, "\mu^{-1}"'] & \mathrm{Sp}(B_{j,\delta^{-1}(i)}^{\Gamma_j \cap \Gamma_{\delta^{-1}(i)}}) \arrow[u, "\rho_{j,\delta^{-1}(i)}"'] \\
\mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}}) \arrow[r, "\gamma^{-1}"'] \arrow[uu, bend left=60, "\rho_{i,\gamma(j)}"] & \mathrm{Sp}(B_{j,\gamma^{-1}(i)}^{\Gamma_j \cap \Gamma_{\gamma^{-1}(i)}}). \arrow[u, "\nu^{-1}"'] \arrow[uu, bend right=60, "\rho_{j,\gamma^{-1}(i)}"']
\end{tikzcd}$$

This yields $\theta_{i,j,\delta} = \theta_{i,j,\gamma}$. □

Since the set $\Gamma_{i,j}$ is finite, the disjoint union of Lemma 6.25 is an admissible open covering of the affinoid variety $Y_{i,j}$. Hence, by (6.3) and Lemma 6.27 we obtain an isomorphism of affinoid varieties

$$\theta_{i,j} = \coprod_{\gamma \in \Gamma_i \backslash \Gamma_{i,j} / \Gamma_j} \theta_{i,j,\gamma} : Y_{i,j} \to Y_{j,i}.$$

For any $i, j, k \in I$, the intersection $Y_{i,j,k} := Y_{i,j} \cap Y_{i,k}$ of two affinoid subdomains of Y_i is an affinoid subdomain of $Y_{i,j}$.

Lemma 6.28. *The system $\{\theta_{i,j}\}_{i,j \in I}$ of isomorphisms satisfies the conditions of* [BGR, Proposition 9.3.2/1]. *Namely,*

(1) $\theta_{i,j} \circ \theta_{j,i} = \mathrm{id}$, $\theta_{i,i} = \mathrm{id}$.

(2) The map $\theta_{i,j}$ induces isomorphisms $\theta_{i,j,k} : Y_{i,j,k} \to Y_{j,i,k}$ satisfying $\theta_{i,j,k} = \theta_{k,j,i} \circ \theta_{i,k,j}$.

Proof. Note that $\{\mathrm{id}\}$ is a complete set of representatives of the coset space $\Gamma_i \backslash \Gamma_{i,i} / \Gamma_i$. Moreover, for any complete set of representatives $I_{i,j}$ of the coset space $\Gamma_i \backslash \Gamma_{i,j} / \Gamma_j$, (6.3) implies that $\{\gamma^{-1} \mid \gamma \in I_{i,j}\}$ is a complete set of representatives of $\Gamma_j \backslash \Gamma_{j,i} / \Gamma_i$. Then Lemma 6.27 and (6.9) yield (1).

Let us prove (2). Since $\theta_{i,j} = \mathrm{id}$ on underlying sets, we have $\theta_{i,j}(Y_{i,j,k}) = \pi(X_i) \cap \pi(X_j) \cap \pi(X_k) = Y_{j,i,k}$. Hence it follows that $\theta_{i,j,k} : Y_{i,j,k} \to Y_{j,i,k}$ is an isomorphism.

Let $\gamma \in \Gamma_{i,j}$. Then we have a covering

$$\pi(X_i \cap X_{\gamma(j)}) \cap \pi(X_k) = \bigcup_{\delta \in \Gamma} \pi(X_i \cap X_{\gamma(j)} \cap X_{\delta(k)}).$$

Note that $\pi(X_i \cap X_{\gamma(j)} \cap X_{\delta(k)}) \neq \varnothing$ only if $\delta \in \Gamma_{i,k}$, and thus this covering has a finite subcovering. Moreover, Lemma 6.24 implies that this is an admissible open covering of the affinoid subdomain $\pi(X_i \cap$

$X_{\gamma(j)}) \cap \pi(X_k) = \pi(X_i \cap X_{\gamma(j)}) \cap Y_{i,k}$ of Y_i. By Lemma 6.25, it is enough to show the cocycle condition on each local piece $\pi(X_i \cap X_{\gamma(j)} \cap X_{\delta(k)})$. By Lemma 6.27, this is the same as showing that the composite

$$\begin{aligned}\theta_{k,j,\delta^{-1}\gamma} \circ \theta_{i,k,\delta} : \pi(X_i \cap X_{\gamma(j)} \cap X_{\delta(k)}) &\to \pi(X_k \cap X_{\delta^{-1}(i)} \cap X_{\delta^{-1}\gamma(j)}) \\ &\to \pi(X_j \cap X_{\gamma^{-1}(i)} \cap X_{\gamma^{-1}\delta(k)})\end{aligned}$$

agrees with $\theta_{i,j,\gamma}$.

By (6.8), we have the commutative diagram

$$\begin{tikzcd}
\pi(X_i \cap X_{\gamma(j)} \cap X_{\delta(k)}) \arrow[r, "\theta_{i,j,\gamma}"] & \pi(X_j \cap X_{\gamma^{-1}(i)} \cap X_{\gamma^{-1}\delta(k)}) \\
\mathrm{Sp}(B_{i,\gamma(j),\delta(k)}^{\Gamma_i \cap \Gamma_{\gamma(j)} \cap \Gamma_{\delta(k)}}) \arrow[u, "\rho_{i,\gamma(j),\delta(k)}", "\wr"'] \arrow[r, "\gamma^{-1}"'] & \mathrm{Sp}(B_{j,\gamma^{-1}(i),\gamma^{-1}\delta(k)}^{\Gamma_j \cap \Gamma_{\gamma^{-1}(i)} \cap \Gamma_{\gamma^{-1}\delta(k)}}). \arrow[u, "\wr", "\rho_{j,\gamma^{-1}(i),\gamma^{-1}\delta(k)}"']
\end{tikzcd}$$

Since the composite

$$\begin{aligned}(\delta^{-1}\gamma)^{-1} \circ \delta^{-1} : X_i \cap X_{\gamma(j)} \cap X_{\delta(k)} &\to X_k \cap X_{\delta^{-1}(i)} \cap X_{\delta^{-1}\gamma(j)} \\ &\to X_j \cap X_{\gamma^{-1}(i)} \cap X_{\gamma^{-1}\delta(k)}\end{aligned}$$

agrees with γ^{-1}, we also have the commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}(B_{i,\gamma(j),\delta(k)}^{\Gamma_i \cap \Gamma_{\gamma(j)} \cap \Gamma_{\delta(k)}}) \arrow[r, "\delta^{-1}"] \arrow[dr, "\gamma^{-1}"'] & \mathrm{Sp}(B_{k,\delta^{-1}(i),\delta^{-1}\gamma(j)}^{\Gamma_k \cap \Gamma_{\delta^{-1}(i)} \cap \Gamma_{\delta^{-1}\gamma(j)}}) \arrow[d, "(\delta^{-1}\gamma)^{-1}"] \\
& \mathrm{Sp}(B_{j,\gamma^{-1}(i),\gamma^{-1}\delta(k)}^{\Gamma_j \cap \Gamma_{\gamma^{-1}(i)} \cap \Gamma_{\gamma^{-1}\delta(k)}}).
\end{tikzcd}$$

Hence we obtain $\theta_{k,j,\delta^{-1}\gamma} \circ \theta_{i,k,\delta} = \theta_{i,j,\gamma}$ on $\pi(X_i \cap X_{\gamma(j)} \cap X_{\delta(k)})$. This concludes the proof. $\square$

Lemma 6.29. *Let $f : \mathrm{Sp}(S) \to \mathrm{Sp}(R)$ be a finite morphism of $\mathbb{K}$-affinoid varieties. Suppose that f is a monomorphism in the category of $\mathbb{K}$-affinoid varieties, in the sense that for any morphisms $g_1, g_2 : Z \to \mathrm{Sp}(S)$ of $\mathbb{K}$-affinoid varieties, the equality $f \circ g_1 = f \circ g_2$ yields $g_1 = g_2$. Then f is a closed immersion.*

Proof. Since R-algebra S is finite, it follows that $S \otimes_R S$ is an affinoid algebra over $\mathbb{K}$. The assumption implies that for any affinoid algebra C over $\mathbb{K}$, any homomorphism $S \otimes_R S \to C$ of $\mathbb{K}$-algebras factors through the natural map $\mu : S \otimes_R S \to S$ defined by $\mu(a \otimes b) = ab$. In particular, we can find a homomorphism $S \to S \otimes_R S$ of $\mathbb{K}$-algebras which makes the following diagram commutative:

$$\begin{tikzcd} S \otimes_R S \arrow[r, "\mu"] \arrow[d, equal] & S \arrow[dl] \\ S \otimes_R S. & \end{tikzcd}$$

Since μ is surjective, this shows that μ is an isomorphism. Now [Sta, Lemma 10.107.1] implies that the finite map $R \to S$ is an epimorphism in the category of rings. By [Sta, Lemma 10.107.6], the map $R \to S$ is surjective. This concludes the proof. □

Proposition 6.30. *Let X be a separated rigid analytic variety over $\mathbb{K}$ equipped with a discrete action of a group Γ. Then there exists a separated rigid analytic variety $\Gamma\backslash X$ over $\mathbb{K}$ and a Γ-invariant morphism $\pi : X \to \Gamma\backslash X$ which satisfies the following universal property: for any separated rigid analytic variety Z over $\mathbb{K}$ and a Γ-invariant morphism $\phi : X \to Z$, there exists a unique morphism $\psi : \Gamma\backslash X \to Z$ satisfying $\phi = \psi \circ \pi$. In particular, such a pair $(\Gamma\backslash X, \pi)$ is unique up to a unique isomorphism.*

Proof. Let Y be the quotient set $\Gamma\backslash X$ and let $\pi : X \to Y = \Gamma\backslash X$ be the quotient map. Put $Y_i = \pi(X_i)$ and $Y_{i,j} = Y_i \cap Y_j$ as before. By Lemma 6.28 and [BGR, Proposition 9.3.2/1], there exists a structure of a rigid analytic variety on Y such that $Y = \bigcup_{i\in I} Y_i$ is an admissible open covering with an isomorphism $\mathrm{Sp}(B_i^{\Gamma_i}) \simeq Y_i$. Since we have a commutative diagram of affinoid varieties

$$\begin{tikzcd} X_i \cap X_j \arrow[r] \arrow[d, equal] & \pi(X_i \cap X_j) \arrow[d, "\theta_{i,j,\mathrm{id}}=\mathrm{id}"] \\ X_j \cap X_i \arrow[r] & \pi(X_j \cap X_i), \end{tikzcd}$$

[BGR, Proposition 9.3.3/1] implies that the natural maps

$$X_i = \mathrm{Sp}(B_i) \to \mathrm{Sp}(B_i^{\Gamma_i}) \xrightarrow{\rho_i} \pi(X_i)$$

glue to define a morphism $\pi : X \to Y$ of rigid analytic varieties over $\mathbb{K}$ whose underlying map is the quotient map.

For any $i \in I$ and $\gamma \in \Gamma$, we have $\Gamma_{\gamma(i),i} = \gamma\Gamma_i$ and $B_{i,i}^{\Gamma_i \cap \Gamma_i} = B_i^{\Gamma_i}$. Thus we have the commutative diagram of affinoid varieties

$$\begin{tikzcd} X_{\gamma(i)} \arrow[r] \arrow[d, "\gamma^{-1}"'] & \mathrm{Sp}(B_{\gamma(i)}^{\Gamma_{\gamma(i)}}) \arrow[r, "\rho_{\gamma(i)}"] \arrow[d, "\gamma^{-1}"'] & \pi(X_{\gamma(i)}) \arrow[d, "\theta_{\gamma(i),i,\gamma}"] \\ X_i \arrow[r] & \mathrm{Sp}(B_i^{\Gamma_i}) \arrow[r, "\rho_i"'] & \pi(X_i). \end{tikzcd}$$

By the construction of Y, this means that π is Γ-invariant.

Let us show the universality. Since $\{Y_i\}_{i \in I}$ is an admissible open covering, by [BGR, Proposition 9.3.3/1] the uniqueness of the morphism ψ can be checked on Y_i, and in this case it follows from Lemma 6.20. For the existence, by Lemma 6.20 there exists a morphism $\psi_i : Y_i \to Z$ satisfying $\phi|_{X_i} = \psi_i \circ \pi|_{X_i}$. For any $i, j \in I$ and $\gamma \in \Gamma_{i,j}$, we have $\phi|_{X_i \cap X_{\gamma(j)}} = \phi|_{X_j \cap X_{\gamma^{-1}(i)}} \circ \gamma^{-1}$ and this induces a commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}}) \arrow[r, hook] \arrow[d, "\gamma^{-1}"'] & Y_{i,j} \arrow[r, "\psi_i|_{Y_{i,j}}"] & Z \arrow[d, equal] \\ \mathrm{Sp}(B_{j,\gamma^{-1}(i)}^{\Gamma_j \cap \Gamma_{\gamma^{-1}(i)}}) \arrow[r, hook] & Y_{j,i} \arrow[r, "\psi_j|_{Y_{j,i}}"'] & Z. \end{tikzcd}$$

Thus we obtain $\psi_i|_{Y_{i,j}} = \psi_j|_{Y_{j,i}} \circ \theta_{i,j}$ and we can find a morphism $\psi : Y \to Z$ by gluing.

Finally, we show that Y is separated over $\mathbb{K}$. For this, it is enough to show that for any $i, j \in I$, the natural map $Y_i \cap Y_j \to Y_i \times_{\mathbb{K}} Y_j$ is a closed immersion. By Lemma 6.25, we see that $Y_i \cap Y_j = Y_{i,j}$ is affinoid. Since this map is a monomorphism of affinoid varieties over $\mathbb{K}$, by Lemma 6.29 we reduce ourselves to showing that this map is finite. Moreover, by Lemma 6.25 and the definition of $\theta_{i,j}$, this map is defined, up to a unique isomorphism, by the ring homomorphism

$$B_i^{\Gamma_i} \hat{\otimes}_{\mathbb{K}} B_j^{\Gamma_j} \to \prod_{\gamma \in \Gamma_i \backslash \Gamma_{i,j} / \Gamma_j} B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}}, \quad b_i \hat{\otimes} b_j \mapsto (b_i \cdot b_j|_{\gamma^{-1}})_\gamma,$$

where the map $b \mapsto b|_{\gamma^{-1}}$ is the one induced by the affinoid map $\gamma^{-1} : X_{\gamma(j)} \to X_j$. Thus it is enough to show that for any $i, j \in I$ and $\gamma \in \Gamma_{i,j}$, the ring homomorphism $B_i^{\Gamma_i} \hat{\otimes}_{\mathbb{K}} B_j^{\Gamma_j} \to B_{i,\gamma(j)}^{\Gamma_i \cap \Gamma_{\gamma(j)}}$ on each factor is finite.

Note that we have a commutative diagram of affinoid algebras

$$\begin{tikzcd} B_i\hat{\otimes}_{\mathbb{K}}B_j \arrow[r, two heads] & B_{i,\gamma(j)} \\ B_i^{\Gamma_i}\hat{\otimes}_{\mathbb{K}}B_j^{\Gamma_j} \arrow[u] \arrow[r] & B_{i,\gamma(j)}^{\Gamma_i\cap\Gamma_{\gamma(j)}}. \arrow[u, hook] \end{tikzcd}$$

Since the assumption that X is separated implies that the map

$$X_i \cap X_{\gamma(j)} \to X_i \times_{\mathbb{K}} X_{\gamma(j)} \stackrel{1\times\gamma^{-1}}{\to} X_i \times_{\mathbb{K}} X_j$$

is a closed immersion, the upper horizontal arrow of the diagram is surjective. Moreover, the map $B_i^{\Gamma_i}\hat{\otimes}_{\mathbb{K}}B_j^{\Gamma_j} \to B_i\hat{\otimes}_{\mathbb{K}}B_j$ is finite. Since $B_{i,\gamma(j)}^{\Gamma_i\cap\Gamma_{\gamma(j)}}$ is a subring of $B_{i,\gamma(j)}$ and the affinoid algebra $B_i^{\Gamma_i}\hat{\otimes}_{\mathbb{K}}B_j^{\Gamma_j}$ is Noetherian, we see that the $B_i^{\Gamma_i}\hat{\otimes}_{\mathbb{K}}B_j^{\Gamma_j}$-algebra $B_{i,\gamma(j)}^{\Gamma_i\cap\Gamma_{\gamma(j)}}$ is finite. This concludes the proof of the proposition. $\square$

6.3. **Carlitz exponential as a uniformizer at ∞.** For any $r, s \in \mathbb{Q}$, consider admissible open subsets Ω_r and $\Omega_{r,s}$ of $\mathbb{P}^1_{\mathbb{C}_\infty}$ as in Definition 5.19.

Lemma 6.31. *For any fractional almost-ideal $\mathfrak{a} \subseteq K$, the action of $\mathfrak{a}$ on Ω_r given by*

$$\mathfrak{a} \times \Omega_r \to \Omega_r, \quad (a, z) \mapsto z + a$$

defines a discrete action of $\mathfrak{a}$ on the separated rigid analytic space Ω_r with respect to the admissible affinoid open covering

$$\{a + \Omega_{r,s} \mid (s, a) \in \mathbb{Z} \times \mathfrak{a}\}.$$

In particular, for any $(s, a) \in \mathbb{Z} \times \mathfrak{a}$, the subset

$$\mathfrak{a}_{(s,a)} = \{b \in \mathfrak{a} \mid b + a + \Omega_{r,s} = a + \Omega_{r,s}\}$$

is equal to the finite group $\mathfrak{a}^{\leqslant q^s}$.

Proof. Note that $z \mapsto z + a$ defines an automorphism of the rigid analytic variety $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_\infty}$. Since $|z + a|_i = |z|_i$ for any $z \in \mathbb{C}_\infty$ and $a \in \mathfrak{a}$, it defines an automorphism on the open subvariety Ω_r. Hence the subset $a + \Omega_{r,s}$ is an admissible affinoid open subset of Ω_r. Moreover, Lemma 5.22 implies that the covering of the lemma is admissible.

To show that the action is discrete, we need to check the conditions of Definition 6.21. The condition (1) follows from the definition. For (2), take any $s \in \mathbb{Z}$. Since $\mathfrak{a}_{(s,a)} = \mathfrak{a}_{(s,0)}$, we may assume $a = 0$. Suppose $b \in \mathfrak{a}_{(s,0)}$. Then for any $z \in \Omega_{r,s}$, we have $|z+b| \leqslant q^s$. Since $|z| \leqslant q^s$, this yields $|b| \leqslant q^s$ and $b \in \mathfrak{a}^{\leqslant q^s}$. Conversely, if $b \in \mathfrak{a}^{\leqslant q^s}$, then for any $z \in \Omega_{r,s}$ we have $|z + b|_i = |z|_i \geqslant q^{-r}$ and $|z + b| \leqslant q^s$. Thus $b + \Omega_{r,s} \subseteq \Omega_{r,s}$.

Since $-b \in \mathfrak{a}^{\leqslant q^s}$, this also yields the reverse containment and $b \in \mathfrak{a}_{(s,0)}$. Hence we obtain $\mathfrak{a}_{(s,0)} = \mathfrak{a}^{\leqslant q^s}$, which is finite by Lemma 6.4.

For the condition (3), first suppose that $b \in \mathfrak{a}$ satisfies $(b+a+\Omega_{r,s}) \cap (a+\Omega_{r,s}) \neq \varnothing$. Then for some $z \in \Omega_{r,s}$ we have $b+z \in \Omega_{r,s}$, which yields $|b| \leqslant q^s$ and $b \in \mathfrak{a}^{\leqslant q^s} = \mathfrak{a}_{(s,a)}$. Next for any $(s,a), (s',a') \in \mathbb{Z} \times \mathfrak{a}$, suppose that $b \in \mathfrak{a}$ satisfies $(a+\Omega_{r,s}) \cap (b+a'+\Omega_{r,s'}) \neq \varnothing$. Then for some $z \in \Omega_{r,s}$ and $z' \in \Omega_{r,s'}$ we have $a+z = b+a'+z'$. This shows that $b+a'-a \in \mathfrak{a}$ satisfies $|b+a'-a| \leqslant q^{s_0}$ with $s_0 = \max\{s,s'\}$ and thus b lies in the finite set $a-a'+\mathfrak{a}^{\leqslant q^{s_0}}$.

Finally, let us check the condition (4). Take any $(s,a) \in \mathbb{Z} \times \mathfrak{a}$. We need to show that the family

$$\{b+a+\Omega_{r,s} \mid b \in \mathfrak{a}\}$$

forms an admissible covering of an admissible open subset of Ω_r. Since the map $z \mapsto z+a$ defines an automorphism on Ω_r, we may assume $a=0$. By Lemma 5.22, it is enough to show that for any $s' \in \mathbb{Z}$, the family

$$\{\Omega_{r,s'} \cap (b+\Omega_{r,s}) \mid b \in \mathfrak{a}\}$$

forms an admissible covering of an admissible open subset of $\Omega_{r,s'}$. By the condition (3), this covering has a finite subcovering. Since the subset

$$\Omega_{r,s'} \cap (b+\Omega_{r,s}) = \{z \in \Omega_{r,s'} \mid |z-b| \leqslant q^s\}$$

is a rational subdomain of the affinoid variety $\Omega_{r,s'}$, the family above has the desired property. $\square$

Since Ω_r is separated, Proposition 6.30 and Lemma 6.31 allow us to define a structure of a separated rigid analytic variety over $\mathbb{C}_\infty$ on the set-theoretic quotient $\mathfrak{a}\backslash\Omega_r$ such that the natural surjection $\pi : \Omega_r \to \mathfrak{a}\backslash\Omega_r$ is a morphism of rigid analytic varieties over $\mathbb{C}_\infty$. By construction, we have $\pi(\Omega_{r,s}) \simeq \mathrm{Sp}(\mathcal{O}(\Omega_{r,s})^{\mathfrak{a}^{\leqslant q^s}})$ and the covering

$$\mathfrak{a}\backslash\Omega_r = \bigcup_{s\in\mathbb{Z}} \pi(\Omega_{r,s})$$

is an admissible affinoid open covering.

Since Ω_r is reduced, the ring $\mathcal{O}(\Omega_{r,s})^{\mathfrak{a}^{\leqslant q^s}}$ is reduced and $\mathfrak{a}\backslash\Omega_r$ is also reduced. Moreover, the universal property of Proposition 6.30 shows that any rigid analytic function f on Ω_r which is fixed by the action of $\mathfrak{a}$ defines a rigid analytic function $\bar{f}$ on $\mathfrak{a}\backslash\Omega_r$ satisfying $\pi^*\bar{f} = f$. Thus we obtain a morphism of rigid analytic varieties over $\mathbb{C}_\infty$

$$\overline{\exp}_{\mathfrak{a}} : \mathfrak{a}\backslash\Omega_r \to \mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_\infty}.$$

Lemma 6.32. *Let $\mathfrak{a}$ be any proper fractional almost-ideal of K and let $N_{\mathfrak{a}} \in \mathbb{Z}$ be as in Lemma 6.2. Let $r \in \mathbb{Z}$ be any integer satisfying $r \leqslant N_{\mathfrak{a}}$. For the map $\exp_{\mathfrak{a}} : \mathbb{C}_\infty \to \mathbb{C}_\infty$, we have*

$$\Omega_r = \exp_{\mathfrak{a}}^{-1}(\{z \in \mathbb{C}_\infty \mid |z| \geqslant \sigma_{\mathfrak{a},q^{-r}}\}).$$

Proof. By Corollary 6.13, we have $\exp_{\mathfrak{a}}(D^\circ_{\mathbb{C}_\infty}(0, q^{-r})) = D^\circ_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})$. Since $\exp_{\mathfrak{a}}$ is additive and $\mathrm{Ker}(\exp_{\mathfrak{a}}) = \mathfrak{a}$, this yields

$$\begin{aligned}\exp_{\mathfrak{a}}^{-1}(D^\circ_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})) &= \mathfrak{a} + D^\circ_{\mathbb{C}_\infty}(0, q^{-r}) \\ &= \bigcup_{a \in \mathfrak{a}} D^\circ_{\mathbb{C}_\infty}(a, q^{-r}) \\ &= \bigcup_{x \in K_\infty} D^\circ_{\mathbb{C}_\infty}(x, q^{-r}),\end{aligned}$$

where the last equality follows from Lemma 6.2. By Lemma 5.17, the latter set is $\Omega \backslash \Omega_r$ and by taking the complement the lemma follows. □

Lemma 6.32 implies that for any proper fractional almost-ideal $\mathfrak{a}$ of K and any $r \in \mathbb{Z}_{\leqslant N_{\mathfrak{a}}}$, the morphism $\overline{\exp}_{\mathfrak{a}}$ factors through the open subvariety

$$D'_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})^\times = \{z \in \mathbb{C}_\infty \mid |z| \geqslant \sigma_{\mathfrak{a},q^{-r}}\}$$

and the resulting morphism $\overline{\exp}_{\mathfrak{a}} : \mathfrak{a}\backslash\Omega_r \to D'_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})^\times$ is bijective. We also have the commutative diagram of rigid analytic varieties

$$\begin{tikzcd} \Omega_r \arrow[d, "\pi"'] \arrow[dr, "\exp_{\mathfrak{a}}"] & \\ \mathfrak{a}\backslash\Omega_r \arrow[r, "\overline{\exp}_{\mathfrak{a}}"'] & D'_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})^\times. \end{tikzcd}$$

Lemma 6.33. *Let $\mathfrak{a}$ be any proper fractional almost-ideal of K. For any $r \in \mathbb{Z}_{\leqslant N_{\mathfrak{a}}}$, the morphism $\overline{\exp}_{\mathfrak{a}} : \mathfrak{a}\backslash\Omega_r \to D'_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})^\times$ is quasi-compact.*

Proof. By Lemma 6.12, the function $r \mapsto \sigma_{\mathfrak{a},q^{-r}}$ is decreasing. For any $t \in \mathbb{Z}_{<r}$, put

$$A_{\mathbb{C}_\infty}(0, [\sigma_{\mathfrak{a},q^{-r}}, \sigma_{\mathfrak{a},q^{-t}}]) = \{z \in \mathbb{C}_\infty \mid \sigma_{\mathfrak{a},q^{-r}} \leqslant |z| \leqslant \sigma_{\mathfrak{a},q^{-t}}\}.$$

Then $\{A_{\mathbb{C}_\infty}(0, [\sigma_{\mathfrak{a},q^{-r}}, \sigma_{\mathfrak{a},q^{-t}}])\}_{t \in \mathbb{Z}_{<r}}$ is an admissible open covering of $D'_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})^\times$.

It is enough to show that $\overline{\exp}_{\mathfrak{a}}^{-1}(A_{\mathbb{C}_\infty}(0, [\sigma_{\mathfrak{a},q^{-r}}, \sigma_{\mathfrak{a},q^{-t}}]))$ is affinoid. By Corollary 6.13 and Lemma 6.32, we have

$$\begin{aligned}\exp_{\mathfrak{a}}^{-1}(A_{\mathbb{C}_\infty}(0, [\sigma_{\mathfrak{a},q^{-r}}, \sigma_{\mathfrak{a},q^{-t}}])) &= \exp_{\mathfrak{a}}^{-1}(D_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-t}}) \cap D'_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{a},q^{-r}})^\times) \\ &= (\mathfrak{a} + D_{\mathbb{C}_\infty}(0, q^{-t})) \cap \Omega_r.\end{aligned}$$

Since π is surjective, we obtain

$$\begin{aligned}\overline{\exp}_{\mathfrak{a}}^{-1}(A_{\mathbb{C}_\infty}(0,[\sigma_{\mathfrak{a},q^{-r}},\sigma_{\mathfrak{a},q^{-t}}])) &= \pi(\exp_{\mathfrak{a}}^{-1}(A_{\mathbb{C}_\infty}(0,[\sigma_{\mathfrak{a},q^{-r}},\sigma_{\mathfrak{a},q^{-t}}]))) \\ &= \pi((\mathfrak{a}+D_{\mathbb{C}_\infty}(0,q^{-t}))\cap\Omega_r) \\ &= \pi(D_{\mathbb{C}_\infty}(0,q^{-t})\cap\Omega_r) \\ &= \pi(\Omega_{r,-t}) = \mathrm{Sp}(\mathcal{O}(\Omega_{r,-t})^{\mathfrak{a}^{\leqslant q^{-t}}}),\end{aligned}$$

which is affinoid. □

Lemma 6.34. *For any proper fractional almost-ideal $\mathfrak{a}$ of K and any $r\in\mathbb{Z}_{\leqslant N_{\mathfrak{a}}}$, the morphism $\overline{\exp}_{\mathfrak{a}}:\mathfrak{a}\backslash\Omega_r\to D'_{\mathbb{C}_\infty}(0,\sigma_{\mathfrak{a},q^{-r}})^\times$ is a locally closed immersion* [BGR, §7.3.3].

Proof. Take any $z\in\Omega_r$. By [BGR, Proposition 7.3.3/4], it is enough to show that $\overline{\exp}_{\mathfrak{a}}$ defines an isomorphism between the complete local rings at $\pi(z)$ and $\exp_{\mathfrak{a}}(z)$. Note that π is given locally by taking the quotient of an affinoid variety by a finite group. Since the action of $\mathfrak{a}$ on Ω_r is fixed point free, the map π is etale by [Sta, Lemma 58.12.4]. Since $\mathbb{C}_\infty$ is algebraically closed, each complete local ring of $\mathfrak{a}\backslash\Omega_r$ is strictly Henselian and π defines an isomorphism between the complete local rings. Thus we are reduced to showing that $\exp_{\mathfrak{a}}$ is a locally closed immersion.

Let $w=\exp_{\mathfrak{a}}(z)$. Then $\exp_{\mathfrak{a}}$ defines a homomorphism of $\mathbb{C}_\infty$-algebras

$$\exp_{\mathfrak{a}}^*:\hat{\mathcal{O}}_{\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_\infty},w}\simeq\mathbb{C}_\infty[[X-w]]\to\hat{\mathcal{O}}_{\mathbb{A}^{1,\mathrm{an}}_{\mathbb{C}_\infty},z}\simeq\mathbb{C}_\infty[[X-z]]$$

which is given by $\exp_{\mathfrak{a}}^*(X)=\exp_{\mathfrak{a}}(X)$. Hence we have

$$\exp_{\mathfrak{a}}^*(X-w)=\exp_{\mathfrak{a}}^*(X)-w=\exp_{\mathfrak{a}}(X)-\exp_{\mathfrak{a}}(z)=\exp_{\mathfrak{a}}(X-z).$$

Since the linear term of $\exp_{\mathfrak{a}}(X)$ is nonzero, the map $\exp_{\mathfrak{a}}^*$ is an isomorphism. □

Proposition 6.35. *For any proper fractional almost-ideal $\mathfrak{a}$ of K and any $r\in\mathbb{Z}_{\leqslant N_{\mathfrak{a}}}$, the morphism $\overline{\exp}_{\mathfrak{a}}:\mathfrak{a}\backslash\Omega_r\to D'_{\mathbb{C}_\infty}(0,\sigma_{\mathfrak{a},q^{-r}})^\times$ is an isomorphism.*

Proof. We know that the map $\overline{\exp}_{\mathfrak{a}}$ is a bijection. By Lemma 6.33 and Lemma 6.34, it is also a quasi-compact locally closed immersion. Then [BGR, Proposition 9.5.3/5] implies that it is a closed immersion. Since it is bijective and the target is reduced, it is an isomorphism. □

Definition 6.36. For any proper fractional almost-ideal $\mathfrak{a}$ of K and any $z\in\Omega$, we define

$$u_{\mathfrak{a}}(z)=\frac{1}{\bar{\pi}\exp_{\mathfrak{a}}(z)}.$$

By Proposition 6.35, for any $r \in \mathbb{Z}_{\leqslant N_{\mathfrak{a}}}$ it defines an isomorphism of rigid analytic varieties

$$\mathfrak{a}\backslash\Omega_r \to D_{\mathbb{C}_\infty}(0, |\bar{\pi}|^{-1}\sigma_{\mathfrak{a},q^{-r}}^{-1})\backslash\{0\}, \quad z \mapsto u_{\mathfrak{a}}(z). \tag{6.10}$$

Lemma 6.37. *Let $a \neq 0 \in A$ and let $m = \deg(a)$. Write*

$$\Phi_a^C(X) = c_0 X + c_1 X^q + \cdots + c_m X^{q^m}, \quad c_i \in A.$$

Then $\deg(c_i) = (m-i)q^i$.

Proof. We proceed by induction on m. If $m = 0$, then $a \in \mathbb{F}_q^\times$ and $\Phi_a^C(X) = aX$, from which the lemma follows for this case.

Suppose that the lemma holds for some $m \geqslant 0$ and $a \in A$ has degree $m+1$. For any $\lambda \in \mathbb{F}_q^\times$, we have $\Phi_{\lambda a}^C(X) = \lambda \Phi_a^C(X)$ and thus we may assume that a is monic. Write $a = tb + \lambda$, where $b \in A$ has degree m and $\lambda \in \mathbb{F}_q$. By the induction hypothesis, we can write

$$\Phi_b^C(X) = c_0 X + c_1 X^q + \cdots + c_m X^{q^m}, \quad \deg(c_i) = (m-i)q^i.$$

Then we have

$$\begin{aligned}\Phi_a^C(X) &= \lambda X + c_0(tX + X^q) + c_1(tX + X^q)^q + \cdots + c_m(tX + X^q)^{q^m} \\ &= (\lambda + tc_0)X + (c_0 + t^q c_1)X^q + \cdots + (c_{i-1} + c_i t^{q^i})X^{q^i} + \cdots + X^{q^{m+1}}.\end{aligned}$$

Since $\deg(c_{i-1}) = (m-i+1)q^{i-1} < (m-i+1)q^i = \deg(c_i t^{q^i})$, the degree of the coefficient of X^{q^i} is $(m-i+1)q^i$. This concludes the proof. $\square$

Definition 6.38. Let $a \neq 0 \in A$ and let $m = \deg(a)$. Write

$$\Phi_a^C(X) = aX + c_1 X^q + \cdots + c_m X^{q^m}, \quad c_m \in \mathbb{F}_q^\times.$$

Then we define

$$f_a(X) := \frac{1}{\Phi_a^C\left(\frac{1}{X}\right)} = \frac{X^{q^m}}{c_m + c_{m-1}X^{q^m - q^{m-1}} + \cdots + aX^{q^m - 1}}.$$

Lemma 6.39. *For any $a \neq 0 \in A$, we have*

$$u_A(az) = f_a(u_A(z)) \quad \textit{for any } z \in \Omega.$$

Proof. Proposition 6.17 yields

$$\begin{aligned}u_A(az) &= \frac{1}{\bar{\pi}\exp_A(az)} = \frac{1}{\Phi_a^C(\bar{\pi}\exp_A(z))} = \frac{1}{\Phi_a^C\left(\frac{1}{u_A(z)}\right)} \\ &= \frac{1}{\frac{c_m}{u_A(z)^{q^m}} + \frac{c_{m-1}}{u_A(z)^{q^{m-1}}} + \cdots + \frac{a}{u_A(z)}} \\ &= \frac{u_A(z)^{q^m}}{c_m + c_{m-1}u_A(z)^{q^m - q^{m-1}} + \cdots + au_A(z)^{q^m - 1}} = f_a(u_A(z)).\end{aligned}$$

$\square$

Lemma 6.40. *Let $a \neq 0 \in A$ and let $m = \deg(a)$. Let ρ be an element of $q^{\mathbb{Q}}$ satisfying $\rho < q^{-1}$. Consider the closed disc*

$$D_{\mathbb{C}_\infty}(0,\rho) = \{u \in \mathbb{C}_\infty \mid |u| \leqslant \rho\}.$$

Then $f_a(u) \in u^{q^m}\mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho))$ and

$$\left|\frac{f_a(u)}{u^{q^m}}\right| = 1 \quad \textit{for any } u \in D_{\mathbb{C}_\infty}(0,\rho).$$

Proof. Write

$$\Phi_a^C(X) = c_0X + c_1X^q + \cdots + c_mX^{q^m}$$

and $\rho = q^r$ with some $r \in \mathbb{Q}$. For any $u \in D_{\mathbb{C}_\infty}(0,\rho)$ and $i < m$, Lemma 6.37 implies

$$(6.11) \qquad |c_iu^{q^m-q^i}| \leqslant q^{(m-i)q^i + r(q^m-q^i)} < 1 \quad \Leftrightarrow \quad r < -\frac{m-i}{q^{m-i}-1}.$$

Let $f(x) = \frac{x}{q^x-1}$, so that $f'(x) = \frac{q^x(1-x\log q)-1}{(q^x-1)^2}$. If $q \geqslant 3$, then $f'(x) < 0$ for any $x \geqslant 1$ and

$$\max\{f(x) \mid x \in \mathbb{Z}_{\geqslant 1}\} = f(1) = \frac{1}{q-1}.$$

If $q = 2$, then $f'(x) < 0$ for any $x \geqslant 2$ and

$$\max\{f(x) \mid x \in \mathbb{Z}_{\geqslant 1}\} = \max\{f(1), f(2)\} = 1.$$

Therefore, if $r < -1$ then the condition (6.11) is satisfied for any integers $m \geqslant 0$ and $i < m$. By the maximum modulus principle, we obtain

$$|c_{m-1}u^{q^m-q^{m-1}} + \cdots + au^{q^m-1}|_{\sup} < 1 \quad \text{on } D_{\mathbb{C}_\infty}(0,\rho).$$

Since $c_m \in \mathbb{F}_q^\times$, the lemma follows. $\square$

6.4. **Uniformizers at cusps.**

Definition 6.41. Let Γ be any arithmetic subgroup of $GL_2(K)$ and $U(K) = \left\{\begin{pmatrix}1 & K\\ 0 & 1\end{pmatrix}\right\}$. Let

$$\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty), \quad \Gamma_\infty^u = \Gamma \cap U(K), \quad \mathfrak{b}_{\Gamma,\infty} = \left\{x \in K \;\middle|\; \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \in \Gamma_\infty\right\}$$

so that we have

$$\Gamma_\infty^u = \begin{pmatrix}1 & \mathfrak{b}_{\Gamma,\infty}\\ 0 & 1\end{pmatrix}.$$

Lemma 6.42. *Let Γ be any arithmetic subgroup of $GL_2(K)$. Then we have*
$$\Gamma_\infty \subseteq \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2(K) \;\middle|\; a, d \in \mathbb{F}_q^\times \right\}.$$

Proof. Take any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty$. The condition $\gamma(\infty) = \infty$ implies $c = 0$.

On the other hand, by Lemma 3.4 there exists a nonzero ideal $\mathfrak{n}$ of A such that $\Gamma(\mathfrak{n}) \subseteq \Gamma$ is a subgroup of finite index. Then $\gamma^n \in \Gamma(\mathfrak{n}) \cap \Gamma_\infty = \Gamma(\mathfrak{n})_\infty$ for some positive integer n. Then a^n and d^n lie in $A^\times = \mathbb{F}_q^\times$. Since $\mathbb{F}_q$ is algebraically closed in K, this yields $a, d \in \mathbb{F}_q^\times$. □

Definition 6.43. Let Γ be an arithmetic subgroup of $GL_2(K)$. By Lemma 6.42, we have the homomorphism
$$\delta_{\Gamma,\infty} : \Gamma_\infty \to \mathbb{F}_q^\times, \quad \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto ad^{-1}.$$
We denote by $w(\Gamma)$ the order of $\mathrm{Im}(\delta_{\Gamma,\infty})$.

Lemma 6.44. *Let Γ be any arithmetic subgroup of $GL_2(K)$. Then $\mathfrak{b}_{\Gamma,\infty}$ is a proper fractional almost-ideal of K which is stable under the multiplication of any element of* $\mathrm{Im}(\delta_{\Gamma,\infty}) \subseteq \mathbb{F}_q^\times$.

Proof. Since $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}$, we see that $\mathfrak{b}_{\Gamma,\infty}$ is an additive subgroup of K. By Lemma 3.4, for some nonzero ideal $\mathfrak{n}$ of A the arithmetic subgroup Γ contains $\Gamma(\mathfrak{n})$ as a subgroup of finite index. Since we have $\mathfrak{b}_{\Gamma(\mathfrak{n}),\infty} = \mathfrak{n}$ and for $U(K) = \left\{\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}\right\}$ the natural map
$$\Gamma \cap U(K)/\Gamma(\mathfrak{n}) \cap U(K) \to \Gamma/\Gamma(\mathfrak{n})$$
is injective, we see that $\mathfrak{n}$ is a subgroup of finite index of $\mathfrak{b}_{\Gamma,\infty}$ and thus the latter is a proper fractional almost-ideal.

Finally, the equality
$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & ad^{-1}x \\ 0 & 1 \end{pmatrix}$$
implies that $\mathfrak{b}_{\Gamma,\infty}$ is stable under $\mathrm{Im}(\delta_{\Gamma,\infty})$. □

Lemma 6.45. *We have*
$$\mathrm{Ker}(\delta_{\Gamma,\infty}) = (Z(\mathbb{F}_q) \cap \Gamma)\Gamma_\infty^u,$$
where $Z(\mathbb{F}_q)$ is the center of $GL_2(A)$.

Proof. The group on the right-hand side lies in $\mathrm{Ker}(\delta_{\Gamma,\infty})$. Conversely, take any $\gamma \in \mathrm{Ker}(\delta_{\Gamma,\infty})$, which can be written as $\gamma = \begin{pmatrix} a & ab \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with some $a \in \mathbb{F}_q^\times$ and $b \in K$. Then $\gamma^q = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in Z(\mathbb{F}_q) \cap \Gamma$ and thus $\gamma^{1-q} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty^u$. This concludes the proof. □

Corollary 6.46. *Let* Γ *be an arithmetic subgroup of* $GL_2(K)$. *Let* $\mathfrak{b} = \mathfrak{b}_{\Gamma,\infty}$ *and* $w = w(\Gamma)$. *For any* $r \in \mathbb{Z}_{\leqslant N_\mathfrak{b}}$, *we have isomorphisms of rigid analytic varieties*

$$\Gamma_\infty^u \backslash \Omega_r \to D_{\mathbb{C}_\infty}(0, |\bar{\pi}|^{-1}\sigma_{\mathfrak{b},q^{-r}}^{-1})\backslash\{0\}, \quad z \mapsto u_\mathfrak{b}(z),$$
$$\Gamma_\infty \backslash \Omega_r \to D_{\mathbb{C}_\infty}(0, |\bar{\pi}|^{-w}\sigma_{\mathfrak{b},q^{-r}}^{-w})\backslash\{0\}, \quad z \mapsto u_\mathfrak{b}(z)^w.$$

Proof. Note that we have $\Gamma_\infty^u\backslash\Omega_r = \mathfrak{b}\backslash\Omega_r$ and the first isomorphism follows from (6.10).

By Lemma 6.45, the group $\mathrm{Ker}(\delta_{\Gamma,\infty})$ acts trivially on $\Gamma_\infty^u\backslash\Omega_r$ and thus the rigid analytic variety $\Gamma_\infty\backslash\Omega_r$ is the quotient of $\Gamma_\infty^u\backslash\Omega_r$ by the action of $\mathrm{Im}(\delta_{\Gamma,\infty})$. The latter group is the unique cyclic subgroup of $\mathbb{F}_q^\times$ of order w and by Lemma 6.44 it stabilizes $\mathfrak{b}$. Then the definition of $e_\mathfrak{b}(X)$ yields

$$u_\mathfrak{b}(cX) = c^{-1}u_\mathfrak{b}(X) \quad \text{for any } c \in \mathrm{Im}(\delta_{\Gamma,\infty})$$

and the isomorphism (6.10) induces the action of $\mathrm{Im}(\delta_{\Gamma,\infty})$ on $D_{\mathbb{C}_\infty}(0, \sigma_{\mathfrak{b},q^{-r}}^{-1})\backslash\{0\}$ given by $z \mapsto c^{-1}z$. Thus the quotient by this action is given by

$$D_{\mathbb{C}_\infty}(0, |\bar{\pi}|^{-1}\sigma_{\mathfrak{b},q^{-r}}^{-1})\backslash\{0\} \to D_{\mathbb{C}_\infty}(0, |\bar{\pi}|^{-w}\sigma_{\mathfrak{b},q^{-r}}^{-w})\backslash\{0\}, \quad z \mapsto z^w.$$

This yields the second isomorphism. □

Definition 6.47. Let $\nu \in GL_2(K)$ and let Γ be an arithmetic subgroup of $GL_2(K)$. Consider the proper fractional almost-ideal $\mathfrak{b}_{\nu^{-1}\Gamma\nu,\infty}$ for the arithmetic subgroup $\nu^{-1}\Gamma\nu$. Define

$$u_{\Gamma,\nu}(z) = u_{\mathfrak{b}_{\nu^{-1}\Gamma\nu,\infty}}(z), \quad \tilde{u}_{\Gamma,\nu}(z) = u_{\Gamma,\nu}(z)^{w(\nu^{-1}\Gamma\nu)}.$$

Lemma 6.48. *Let* Γ *be an arithmetic subgroup of* $GL_2(K)$. *Let* $\xi = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in GL_2(K)$. *Then we have*

$$\mathfrak{b}_{\xi^{-1}\Gamma\xi,\infty} = A^{-1}D\mathfrak{b}_{\Gamma,\infty}, \quad w(\xi^{-1}\Gamma\xi) = w(\Gamma),$$
$$u_{\Gamma,\xi}(z) = AD^{-1}u_{\Gamma,\mathrm{id}}(AD^{-1}z), \quad \tilde{u}_{\Gamma,\xi}(z) = (AD^{-1})^{w(\Gamma)}\tilde{u}_{\Gamma,\mathrm{id}}(AD^{-1}z).$$

Proof. Since $\xi(\infty) = \infty$, we have $(\xi^{-1}\Gamma\xi)_\infty = \xi^{-1}\Gamma_\infty\xi$. Moreover, from the equality

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} a & A^{-1}((a-d)B + bD) \\ 0 & d \end{pmatrix},$$

we obtain $\mathfrak{b}_{\xi^{-1}\Gamma\xi,\infty} = A^{-1}D\mathfrak{b}_{\Gamma,\infty}$, $\mathrm{Im}(\delta_{\xi^{-1}\Gamma\xi,\infty}) = \mathrm{Im}(\delta_{\Gamma,\infty})$ and $w(\xi^{-1}\Gamma\xi) = w(\Gamma)$. The first equality yields

$$u_{\mathfrak{b}_{\xi^{-1}\Gamma\xi,\infty}}(z) = AD^{-1}u_{\mathfrak{b}_{\Gamma,\infty}}(AD^{-1}z),$$

from which the lemma follows. □

Lemma 6.49. *Let Γ be an arithmetic subgroup of $GL_2(K)$. Let $\xi \in GL_2(K)$ with $\xi = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Put $x = AD^{-1}$ and $y = BD^{-1}$. Write $|x| = q^m$ and*

$$\mathfrak{b} = \mathfrak{b}_{\Gamma,\infty}, \quad \mathfrak{b}' = \mathfrak{b}_{\xi^{-1}\Gamma\xi,\infty}.$$

Moreover, for any integer r, write

$$\rho_r = |\bar{\pi}|^{-1}\sigma^{-1}_{\mathfrak{b},q^{-r}}, \quad \rho'_r = |\bar{\pi}|^{-1}\sigma^{-1}_{\mathfrak{b}',q^{-r}}.$$

Then, for any sufficiently small r, we have the commutative diagram of isomorphisms of rigid analytic varieties

$$\begin{array}{ccc} \mathfrak{b}_{\xi^{-1}\Gamma\xi,\infty}\backslash\Omega_r & \xrightarrow{u_{\Gamma,\xi}} & D_{\mathbb{C}_\infty}(0,\rho'_r)\backslash\{0\} \\ \Big\downarrow{\scriptstyle\xi} & & \Big\downarrow{\scriptstyle g_\xi} \\ \mathfrak{b}_{\Gamma,\infty}\backslash\Omega_{r-m} & \xrightarrow[u_{\Gamma,\mathrm{id}}]{} & D_{\mathbb{C}_\infty}(0,\rho_{r-m})\backslash\{0\}, \end{array}$$

where g_ξ is the restriction of the isomorphism

$$g_\xi : D_{\mathbb{C}_\infty}(0,\rho'_r) \to D_{\mathbb{C}_\infty}(0,\rho_{r-m}), \quad u \mapsto \frac{x^{-1}u}{1 + \bar{\pi}\exp_{\mathfrak{b}}(u)x^{-1}u}.$$

Proof. By Lemma 6.48, we have $\mathfrak{b}' = x^{-1}\mathfrak{b}$ and the isomorphism

$$\Omega_r \to \Omega_{r-m}, \quad z \mapsto \xi(z)$$

induces the left vertical arrow.

For any $\rho \in q^{\mathbb{Q}}$, this also yields

$$\begin{aligned} \sigma_{\mathfrak{b}',\rho} &= \rho \prod_{b\in x^{-1}\mathfrak{b},\ 0<|b|<\rho} \frac{\rho}{|b|} = \rho \prod_{b\in\mathfrak{b},\ 0<|b|<\rho|x|} \frac{\rho}{|x^{-1}b|} \\ &= \rho \prod_{b\in\mathfrak{b},\ 0<|b|<\rho|x|} \frac{\rho|x|}{|b|} = |x|^{-1}\sigma_{\mathfrak{b},\rho|x|}. \end{aligned}$$

This implies $\rho'_r = |x|\rho_{r-m}$ and we have an isomorphism

$$D_{\mathbb{C}_\infty}(0, \rho'_r) \to D_{\mathbb{C}_\infty}(0, \rho_{r-m}), \quad u \mapsto x^{-1}u.$$

Hence, if r is sufficiently small so that $\rho_{r-m}|\bar{\pi}\exp_{\mathfrak{b}}(y)| < 1$, then we have an isomorphism

$$D_{\mathbb{C}_\infty}(0, \rho_{r-m}) \to D_{\mathbb{C}_\infty}(0, \rho_{r-m}), \quad v \mapsto \frac{v}{1 + \bar{\pi}\exp_{\mathfrak{b}}(y)v}$$

and we obtain the isomorphism

$$g_\xi : D_{\mathbb{C}_\infty}(0, \rho'_r) \to D_{\mathbb{C}_\infty}(0, \rho_{r-m})$$

preserving the origin.

Now Lemma 6.48 yields $u_{\Gamma,\xi}(z) = xu_{\Gamma,\mathrm{id}}(xz)$ and thus

$$\begin{aligned}
u_{\Gamma,\mathrm{id}}(\xi(z)) &= u_{\Gamma,\mathrm{id}}(xz + y) \\
&= \frac{1}{\bar{\pi}\exp_{\mathfrak{b}}(xz + y)} = \frac{1}{\bar{\pi}\exp_{\mathfrak{b}}(xz) + \bar{\pi}\exp_{\mathfrak{b}}(y)} \\
&= \frac{1}{\frac{1}{u_{\Gamma,\mathrm{id}}(xz)} + \bar{\pi}\exp_{\mathfrak{b}}(y)} = \frac{1}{\frac{1}{x^{-1}u_{\Gamma,\xi}(z)} + \bar{\pi}\exp_{\mathfrak{b}}(y)} \\
&= \frac{x^{-1}u_{\Gamma,\xi}(z)}{1 + \bar{\pi}\exp_{\mathfrak{b}}(y)x^{-1}u_{\Gamma,\xi}(z)} = g_\xi(u_{\Gamma,\xi}(z)).
\end{aligned}$$

This concludes the proof. □

Lemma 6.50. *Let Γ be an arithmetic subgroup of $GL_2(K)$ which is p'-torsion free. Then we have $w(\Gamma) = 1$.*

Proof. Take any $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \Gamma_\infty$, so that $a, d \in \mathbb{F}_q^\times$ by Lemma 6.42. If $a \neq d$, then we have

$$\begin{aligned}
\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^q &= \begin{pmatrix} a^q & b(a^{q-1} + a^{q-2}d + \cdots + d^{q-1}) \\ 0 & d^q \end{pmatrix} \\
&= \begin{pmatrix} a & b\left(\frac{a^q - d^q}{a - d}\right) \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}
\end{aligned}$$

and thus $\gamma^{q-1} = \mathrm{id}$, which contradicts the assumption that Γ is p'-torsion free. Hence we obtain $a = d$ and $\mathrm{Im}(\delta_{\Gamma,\infty}) = \{1\}$. □

6.5. **Definition of Drinfeld modular forms.**

Definition 6.51. Let $f : \Omega \to \mathbb{C}_\infty$ be a function on Ω. For any $\xi \in GL_2(K)$ and $k, m \in \mathbb{Z}$, define functions $f|_{k,m}\xi$ and $f|_k\xi$ on Ω by

$$(f|_{k,m}\xi)(z) = \det(\xi)^m(cz + d)^{-k}f(\xi(z)), \quad \xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $f|_k\xi = f|_{k,k-1}\xi$. We call $f \mapsto f|_{k,m}\xi$ and $f \mapsto f|_k\xi$ the slash operators.

Lemma 6.52. *Let ξ_1, ξ_2 be elements of $GL_2(K)$. Then we have*

$$(f|_{k,m}\xi_1)|_{k,m}\xi_2 = f|_{k,m}\xi_1\xi_2.$$

Proof. Write $\xi_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Then we have

$$\begin{aligned}((f|_{k,m}\xi_1)|_{k,m}\xi_2)(z) &= \det(\xi_2)^m(c_2z+d_2)^{-k}(f|_{k,m}\xi_1)(\xi_2(z)) \\ &= \det(\xi_1\xi_2)^m(c_2z+d_2)^{-k}\left(c_1\frac{a_2z+b_2}{c_2z+d_2}+d_1\right)^{-k} f((\xi_1\xi_2)(z)) \\ &= \det(\xi_1\xi_2)^m((c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2))^{-k}f((\xi_1\xi_2)(z)) \\ &= (f|_{k,m}\xi_1\xi_2)(z)\end{aligned}$$

and the lemma follows. □

Lemma 6.53. *For any $k, m \in \mathbb{Z}$, $f \in \mathcal{O}(\Omega)$ and $\xi \in GL_2(K)$, we have*

$$f|_{k,m}\xi \in \mathcal{O}(\Omega).$$

Proof. By Corollary 5.27, the function

$$\Omega \to \mathbb{C}_\infty, \quad z \mapsto f(\xi(z))$$

is analytic. On the other hand, write $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then the function $z \mapsto cz+d$ is analytic and nowhere vanishing on Ω. Thus $cz+d \in \mathcal{O}(\Omega)^\times$ and the lemma follows. □

Let Γ be an arithmetic subgroup of $GL_2(K)$ and $k, m \in \mathbb{Z}$. Let $f : \Omega \to \mathbb{C}_\infty$ be a rigid analytic function on Ω satisfying

$$f(\gamma(z)) = \det(\gamma)^{-m}(cz+d)^k f(z) \quad \text{for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Let $\nu \in GL_2(K)$. Then Lemma 6.53 yields $f|_{k,m}\nu \in \mathcal{O}(\Omega)$. For any $\eta \in (\nu^{-1}\Gamma\nu)^u_\infty$, we have

$$(f|_{k,m}\nu)(\eta(z)) = (f|_{k,m}\nu\eta)(z) = (f|_{k,m}\nu)(z).$$

By Corollary 6.46, we can write

$$(f|_{k,m}\nu)(z) = \sum_{i\in\mathbb{Z}} c_i(f,\nu)u_{\Gamma,\nu}(z)^i, \quad c_i(f,\nu) \in \mathbb{C}_\infty, \tag{6.12}$$

where $\sum_{i\in\mathbb{Z}} c_i(f,\nu)X^i$ converges on a punctured closed disc of some positive radius centered at the origin.

Definition 6.54. Let f and ν be as above. We refer to (6.12) as the Fourier expansion of f for ν. By abuse of language, we also call it the Fourier expansion at the cusp $[\nu(\infty)]$ represented by $\nu(\infty)$, though it depends on the choice of ν.

We say f is regular (*resp.* vanishes *resp.* vanishes twice) at the cusp $[\nu(\infty)]$ if $c_i(f,\nu) = 0$ for any $i < 0$ (*resp.* $i \leqslant 0$ *resp.* $i \leqslant 1$).

Definition 6.55. Let Γ be an arithmetic subgroup of $GL_2(K)$ and $k, m \in \mathbb{Z}$. Let $f : \Omega \to \mathbb{C}_\infty$ be a rigid analytic function on Ω. We say f is a Drinfeld modular form (*resp.* cuspform *resp.* double cuspform) of level Γ, weight k and type m if f satisfies $f|_{k,m}\gamma = f$, namely

$$f(\gamma(z)) = \det(\gamma)^{-m}(cz+d)^k f(z) \quad \text{for any } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{6.13}$$

and f is regular (*resp.* vanishes *resp.* vanishes twice) at the cusp $[\nu(\infty)]$ for any $\nu \in GL_2(K)$.

Lemma 6.56. *For any $f \in \mathcal{O}(\Omega)$ satisfying (6.13) and any $\nu \in GL_2(K)$, the validity of each condition on $c_i(f,\nu)$ for ν in Definition 6.55 depends only on $[\nu(\infty)] \in \Gamma\backslash\mathbb{P}^1(K)$.*

Proof. It is enough to show that if the condition holds for ν, then it holds for $\gamma\nu\xi$ with any $\gamma \in \Gamma$ and $\xi = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in GL_2(K)$. Since $(\gamma\nu\xi)^{-1}\Gamma(\gamma\nu\xi) = (\nu\xi)^{-1}\Gamma(\nu\xi)$ and $f|_{k,m}\gamma = f$, we may assume $\gamma = \mathrm{id}$. Replacing f by $f|_{k,m}\nu$ and Γ by $\nu^{-1}\Gamma\nu$, we may assume $\nu = \mathrm{id}$.

Consider the isomorphism g_ξ of Lemma 6.49. Since we have $(f|_{k,m}\xi)(z) = (AD)^m D^{-k} f(\xi(z))$ and g_ξ preserves the vanishing order at the origin, the lemma follows from the commutative diagram of Lemma 6.49. $\square$

The $\mathbb{C}_\infty$-vector spaces of Drinfeld modular forms (*resp.* cuspforms *resp.* double cuspforms) of level Γ, weight k and type m are denoted by

$$M_{k,m}(\Gamma), \quad S_{k,m}(\Gamma), \quad S^{(2)}_{k,m}(\Gamma).$$

When $m = k-1$, we say a Drinfeld modular form is of level Γ and weight k, and we drop m from the subscripts of the spaces above.

Lemma 6.57. *Let Γ be an arithmetic subgroup of $GL_2(K)$ and $k, m \in \mathbb{Z}$. Then we have $M_{k,m}(\Gamma) = 0$ if $k \not\equiv 2m \bmod |Z(\mathbb{F}_q) \cap \Gamma|$.*

Proof. Take any $\gamma = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in Z(\mathbb{F}_q) \cap \Gamma$ and $f \in M_{k,m}(\Gamma)$. Then we have

$$f(z) = f(\gamma(z)) = a^{k-2m} f(z),$$

from which the lemma follows. $\square$

Lemma 6.58. *Let $\rho \in q^{\mathbb{Q}}$ and $a \in \mathbb{C}_\infty$. Let $f \in \mathcal{O}(D_{\mathbb{C}_\infty}(a,\rho)\backslash\{a\})$. Then f is bounded on $D_{\mathbb{C}_\infty}(a,\rho)\backslash\{a\}$ if and only if f uniquely extends to an element of $\mathcal{O}(D_{\mathbb{C}_\infty}(a,\rho))$.*

Proof. We may assume $a = 0$ and $\rho = 1$. By the maximal modulus principle, the "if" part is clear.

Let us consider the "only if" part. Since $\mathcal{O}(D_{\mathbb{C}_\infty}(0,1)) = \mathbb{C}_\infty\langle x\rangle$ is a PID, its element with infinitely many zeroes is zero. This implies the uniqueness of the extension of f.

For the existence, write $f = \sum_{n\in\mathbb{Z}} a_n x^n$ with $a_n \in \mathbb{C}_\infty$. Let $\mathcal{P}$ be the Newton polygon of f in the x-y plane and put

$$\mathcal{P}_{\leqslant 0} = \mathcal{P} \cap \{(x,y) \in \mathbb{R}^2 \mid x \leqslant 0\}.$$

Let Σ be the set of slopes of $\mathcal{P}_{\leqslant 0}$ and let $s_l \in \mathbb{Q}$ be the l-th largest element of Σ. Taking any $\sigma_l \in \mathbb{Q} \cap (s_{l+1}, s_l)$ for $l < |\Sigma|$ and $\sigma_l \in \mathbb{Q} \cap (-\infty, s_{|\Sigma|})$ otherwise, we can find a sequence $\{\sigma_l\}_{l\geqslant 1}$ in $\mathbb{Q}\backslash\Sigma$ satisfying $\lim_{l\to\infty}\sigma_l = -\infty$. In particular, there exists $L \geqslant 1$ satisfying $\sigma_l \leqslant 0$ for any $l \geqslant L$.

Note that, if $z \in \mathbb{C}_\infty^\times$ satisfies $t = v_\infty(z) \geqslant 0$ and $-t$ is not a slope of $\mathcal{P}$, then $v_\infty(f(z))$ agrees with the y-intercept of the tangent line of $\mathcal{P}$ of slope $-t$. By assumption, there exists $M \in \mathbb{R}$ satisfying $v_\infty(f(z)) \geqslant M$ for any $z \in \mathbb{C}_\infty^\times$ with $v_\infty(z) \geqslant 0$. Then for any $l \geqslant L$, the polygon $\mathcal{P}_{\leqslant 0}$ lies above the line $y = \sigma_l x + M$. Since $\lim_{l\to\infty}\sigma_l = -\infty$, this forces $a_n = 0$ for any $n < 0$ and thus $f \in \mathbb{C}_\infty\langle x\rangle$. ☐

Lemma 6.59. *Let $\rho \in q^{\mathbb{Q}}$ and $a \in \mathbb{C}_\infty$. Let $f \in \mathcal{O}(D_{\mathbb{C}_\infty}(a,\rho))$. Then $f(a) = 0$ if and only if*

$$\lim_{\sigma\to 0} \sup_{z\in D_{\mathbb{C}_\infty}(a,\sigma)\backslash\{a\}} |f(z)| = 0.$$

Proof. We may assume $a = 0$ and $\rho = 1$. Write

$$f = \sum_{n\geqslant 0} a_n x^n, \quad a_n \in \mathbb{C}_\infty$$

with $\lim_{n\to\infty}|a_n| = 0$. If $f(0) = 0$, then $a_0 = 0$ and we can write $f = xg$ with $g \in \mathbb{C}_\infty\langle x\rangle$. Let $|g|_{\sup}$ be the supremum norm of g for $\mathbb{C}_\infty\langle x\rangle$. For any $\sigma \in q^{\mathbb{Q}} \cap (0,1]$ we have

$$\sup_{z\in D_{\mathbb{C}_\infty}(a,\sigma)\backslash\{a\}} |f(z)| \leqslant \sigma|g|_{\sup}$$

and thus the value on the left-hand side goes to zero.

Conversely, suppose that f satisfies the limit condition of the lemma and $a_0 \neq 0$. Then we have $|a_0| > |f(z) - a_0|$ for any $z \neq 0 \in \mathbb{C}_\infty$ with sufficiently small $|z|$. This implies $|f(z)| = |a_0| > 0$, which contradicts the assumption. ☐

Lemma 6.60. *Let Γ be an arithmetic subgroup of $GL_2(K)$, $k, m \in \mathbb{Z}$, $\nu \in GL_2(K)$ and $s = \nu(\infty)$. Let $f \in \mathcal{O}(\Omega)$ satisfying (6.13).*

(1) f is regular at the cusp $[s]$ if and only if

$$\sup_{z \in \Omega_r} |(f|_{k,m}\nu)(z)| < +\infty$$

for some sufficiently small integer r.

(2) f vanishes at the cusp $[s]$ if and only if

$$\lim_{r \to -\infty} \sup_{z \in \Omega_r} |(f|_{k,m}\nu)(z)| = 0.$$

Proof. Let $\mathfrak{b} = \mathfrak{b}_{\nu^{-1}\Gamma\nu,\infty}$ and let r be any integer satisfying $r \leqslant N_{\mathfrak{b}}$. Put

$$\rho = |\bar{\pi}|^{-1}\sigma_{\mathfrak{b},q^{-r}}^{-1}.$$

By Corollary 6.46 we have an isomorphism

$$(\nu^{-1}\Gamma\nu)_\infty^u \backslash \Omega_r \to D_{\mathbb{C}_\infty}(0, \rho)\backslash\{0\}, \quad z \mapsto u = u_{\Gamma,\nu}(z).$$

Put $F = f|_{k,m}\nu$. Let $\bar{F}$ be the rigid analytic function on $(\nu^{-1}\Gamma\nu)_\infty^u\backslash\Omega_r$ that F induces. Since the natural map $\pi : \Omega_r \to (\nu^{-1}\Gamma\nu)_\infty^u\backslash\Omega_r$ is surjective and $F = \pi^*(\bar{F})$, we have

$$\sup_{z \in \Omega_r} |F(z)| = \sup_{z \in (\nu^{-1}\Gamma\nu)_\infty^u\backslash\Omega_r} |\bar{F}(z)|.$$

Then the assertion (1) follows from Lemma 6.58.

Moreover, the inequality $\sigma_{\mathfrak{b},q^{-r}} \geqslant q^{-r}$ of Definition 6.11 implies $\rho \to 0$ when $r \to -\infty$. Then the assertion (2) follows from (1) and Lemma 6.59. $\square$

Lemma 6.61. *Let $\Gamma' \subseteq \Gamma$ be arithmetic subgroups of $GL_2(K)$. Then we have*

$$M_{k,m}(\Gamma) \subseteq M_{k,m}(\Gamma'), \quad S_{k,m}(\Gamma) \subseteq S_{k,m}(\Gamma').$$

Proof. If $f \in \mathcal{O}(\Omega)$ satisfies $f|_{k,m}\gamma = f$ for any $\gamma \in \Gamma$, then it holds for any $\gamma \in \Gamma'$. Then the lemma follows from Lemma 6.60. $\square$

Lemma 6.62. *Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $\nu \in GL_2(K)$. Then the map $f \mapsto f|_{k,m}\nu$ induces isomorphisms*

$$M_{k,m}(\Gamma) \to M_{k,m}(\nu^{-1}\Gamma\nu), \quad S_{k,m}(\Gamma) \to S_{k,m}(\nu^{-1}\Gamma\nu).$$

Proof. For any $f \in M_{k,m}(\Gamma)$, the function $g = f|_{k,m}\nu$ satisfies

$$g|_{k,m}(\nu^{-1}\gamma\nu) = g \quad \text{for any } \gamma \in \Gamma.$$

Moreover, Lemma 6.60 shows that for any $\xi \in GL_2(K)$, the function g is regular (*resp.* vanishes) at the cusp $[\xi(\infty)]$ if and only if

$$\sup_{z \in \Omega_r} |(f|_{k,m}\nu\xi)(z)| < +\infty \quad \textit{resp.} \quad \lim_{r \to -\infty} \sup_{z \in \Omega_r} |(f|_{k,m}\nu\xi)(z)| = 0$$

if and only if f is regular (*resp.* vanishes) at the cusp $[\nu\xi(\infty)]$. This concludes the proof. $\square$

Lemma 6.63. *Let $\Gamma' \lhd \Gamma$ be arithmetic subgroups of $GL_2(K)$. Then the group Γ/Γ' acts on the $\mathbb{C}_\infty$-vector spaces $M_{k,m}(\Gamma')$ and $S_{k,m}(\Gamma')$ by*

$$f \mapsto f|_{k,m}\gamma, \quad \gamma \in \Gamma.$$

Moreover, we have

$$M_{k,m}(\Gamma) = M_{k,m}(\Gamma')^{\Gamma/\Gamma'}, \quad S_{k,m}(\Gamma) = S_{k,m}(\Gamma')^{\Gamma/\Gamma'}.$$

Proof. For any $f \in \mathcal{O}(\Omega)$, we have $f|_{k,m}\gamma = f$ for any $\gamma \in \Gamma$ if and only if it holds for any $\gamma \in \Gamma'$ and f is fixed by the action of Γ/Γ'. Moreover, for any $\gamma \in \Gamma$ we have $\gamma GL_2(K) = GL_2(K)$, and Lemma 6.60 shows that for any $\nu \in GL_2(K)$, the function $f|_{k,m}\gamma$ is regular (*resp.* vanishes) at the cusp $[\nu(\infty)]$ if and only if

$$\sup_{z\in\Omega_r} |(f|_{k,m}\gamma\nu)(z)| < +\infty \quad \textit{resp.} \quad \lim_{r\to-\infty} \sup_{z\in\Omega_r} |(f|_{k,m}\gamma\nu)(z)| = 0$$

if and only if f is regular (*resp.* vanishes) at the cusp $[\gamma\nu(\infty)]$. This concludes the proof. $\square$

7. Operators acting on Drinfeld modular forms

7.1. **Double coset operators.**

Lemma 7.1. *Let Γ_1, Γ_2 be congruence subgroups of $GL_2(A)$. Let $\xi \in GL_2(K)$. Put $\Gamma_3 = \xi^{-1}\Gamma_1\xi \cap \Gamma_2$. Then the map*

$$\Gamma_3\backslash\Gamma_2 \to \Gamma_1\backslash\Gamma_1\xi\Gamma_2, \quad \Gamma_3\gamma_2 \mapsto \Gamma_1\xi\gamma_2$$

is a bijection. Moreover, the coset space $\Gamma_1\backslash\Gamma_1\xi\Gamma_2$ is finite.

Proof. The map of the lemma is well-defined and surjective. For the injectivity, suppose that elements $\gamma_2, \gamma_2' \in \Gamma_2$ satisfy $\Gamma_1\xi\gamma_2 = \Gamma_1\xi\gamma_2'$. Then $\gamma_2' = \xi^{-1}\gamma_1\xi\gamma_2$ with some $\gamma_1 \in \Gamma_1$. Then we have $\xi^{-1}\gamma_1\xi \in \Gamma_2$ and $\gamma_2' \in \Gamma_3\gamma_2$.

Moreover, by Lemma 3.4 there exist nonzero ideals $\mathfrak{n}_1, \mathfrak{n}_2 \subseteq A$ satisfying $\Gamma(\mathfrak{n}_1) \subseteq \xi^{-1}\Gamma_1\xi$ and $\Gamma(\mathfrak{n}_2) \subseteq \Gamma_2$. Then $\Gamma(\mathfrak{n}_1 \cap \mathfrak{n}_2) \subseteq \Gamma_3 \subseteq GL_2(A)$ and thus $[\Gamma_2 : \Gamma_3] < +\infty$, which yields the latter assertion of the lemma. $\square$

Lemma 7.2. *Let Γ_1, Γ_2 be congruence subgroups of $GL_2(A)$. Let $\xi \in GL_2(K)$. Let k, m be integers. Write*

$$\Gamma_1\xi\Gamma_2 = \coprod_{i=1}^{r} \Gamma_1\xi_i.$$

Then we have a $\mathbb{C}_\infty$-linear map

$$M_{k,m}(\Gamma_1) \to M_{k,m}(\Gamma_2), \quad f \mapsto f|_{k,m}[\Gamma_1\xi\Gamma_2] := \sum_{i=1}^{r} f|_{k,m}\xi_i$$

which induces a map $S_{k,m}(\Gamma_1) \to S_{k,m}(\Gamma_2)$.

Proof. Note that the map of the lemma is independent of the choice of ξ_i. For any $\gamma_2 \in \Gamma_2$, we have

$$\Gamma_1\xi_i\gamma_2 \cap \Gamma_1\xi_j\gamma_2 = \varnothing \ (i \neq j), \quad \Gamma_1\xi\Gamma_2 = \coprod_{i=1}^{r} \Gamma_1\xi_i\gamma_2$$

and the set $\{\xi_i\gamma_2\}_i$ is also a complete set of representatives of the coset space $\Gamma_1\backslash\Gamma_1\xi\Gamma_2$. Thus we have

$$(\sum_i f|_{k,m}\xi_i)|_{k,m}\gamma_2 = \sum_i f|_{k,m}\xi_i\gamma_2 = \sum_i f|_{k,m}\xi_i.$$

Now the lemma follows from Lemma 6.60. □

7.2. **Hecke operators.**

Definition 7.3. Let $\mathfrak{n}$ be a nonzero monic element of A and let Θ be a subgroup of $(A/(\mathfrak{n}))^\times$. Define

$$\Gamma_0^\Theta(\mathfrak{n}) := \left\{ \gamma \in SL_2(A) \;\middle|\; \gamma \bmod \mathfrak{n} \in \begin{pmatrix} \Theta & * \\ 0 & \Theta \end{pmatrix} \right\}.$$

For $\Theta = (A/(\mathfrak{n}))^\times$ or $\Theta = \{1\}$, we denote it by

$$\Gamma_0(\mathfrak{n}) := \Gamma_0^{(A/(\mathfrak{n}))^\times}(\mathfrak{n}), \quad \Gamma_1(\mathfrak{n}) = \Gamma_0^{\{1\}}(\mathfrak{n}),$$

so that we have

$$\Gamma_1(\mathfrak{n}) \subseteq \Gamma_0^\Theta(\mathfrak{n}) \subseteq \Gamma_0(\mathfrak{n}).$$

Since they contain $\Gamma(\mathfrak{n})$, they are congruence subgroups of $GL_2(A)$. In particular, for $\mathfrak{n} = 1$ we have $\Gamma_0(\mathfrak{n}) = \Gamma_1(\mathfrak{n}) = SL_2(A)$.

Note that the natural map $SL_2(A) \to SL_2(A/(\mathfrak{n}))$ is surjective.

Lemma 7.4. *Let $\mathfrak{n} \in A_+$ be a nonzero element and let $Q \in A\backslash\mathbb{F}_q$ be any monic irreducible polynomial. Put $\mathbb{F}_Q = A/(Q)$. Let $J(\mathfrak{n}, Q) \subseteq \Gamma_1(\mathfrak{n})$ be any subset such that the map*

$$J(\mathfrak{n}, Q) \to \left\{ \begin{array}{cc} \mathbb{P}^1(\mathbb{F}_Q)\backslash\{(0:1)\} & (Q \mid \mathfrak{n}) \\ \mathbb{P}^1(\mathbb{F}_Q) & (Q \nmid \mathfrak{n}) \end{array} \right\}, \quad \gamma \mapsto (1:0)\gamma$$

is bijective. Then we have

$$\Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n}) = \coprod_{\xi \in I(\mathfrak{n},Q)} \Gamma_0^\Theta(\mathfrak{n})\xi, \quad I(\mathfrak{n}, Q) = \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} J(\mathfrak{n}, Q).$$

Proof. Put $\Gamma = \Gamma_0^\Theta(\mathfrak{n})$. Then

$$\Gamma' := \Gamma \cap \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}^{-1} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; b \equiv 0 \bmod Q \right\}.$$

When $Q \mid \mathfrak{n}$, we have natural bijections

$$\begin{aligned}\Gamma' \backslash \Gamma &\to \left\{ \begin{pmatrix} a & QA/\mathfrak{n}A \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \Theta \right\} \backslash \left\{ \begin{pmatrix} a & A/\mathfrak{n}A \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in \Theta \right\} \\ &\to \mathbb{P}^1(\mathbb{F}_Q) \backslash \{(0:1)\},\end{aligned}$$

where the last map is given by $(1:0) \mapsto (1:0)\gamma$. On the other hand, when $Q \nmid \mathfrak{n}$, we have natural bijections

$$\Gamma' \backslash \Gamma \to \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} \backslash SL_2(\mathbb{F}_Q) \to \mathbb{P}^1(\mathbb{F}_Q), \quad \gamma \mapsto (1:0)\gamma.$$

Thus Lemma 7.2 concludes the proof. □

Example 7.5. Let $\mathfrak{n} \in A_+$ and let $Q \in A \backslash \mathbb{F}_q$ be any monic irreducible polynomial. For any $\beta \in A$, put

$$\eta_\beta := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \Gamma_1(\mathfrak{n}), \quad \xi_\beta := \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_\beta = \begin{pmatrix} 1 & \beta \\ 0 & Q \end{pmatrix}.$$

When $Q \nmid \mathfrak{n}$, we choose $R, S \in A$ satisfying $RQ - \mathfrak{n}S = 1$ and put

$$\eta_\infty := \begin{pmatrix} RQ & S \\ \mathfrak{n} & 1 \end{pmatrix} \in \Gamma_1(\mathfrak{n}), \quad \xi_\infty := \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_\infty = \begin{pmatrix} R & S \\ \mathfrak{n} & Q \end{pmatrix} \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the set

$$I(\mathfrak{n}, Q) = \begin{cases} \{\xi_\beta \mid \deg(\beta) < \deg(Q)\} & (Q \mid \mathfrak{n}), \\ \{\xi_\beta \mid \deg(\beta) < \deg(Q)\} \cup \{\xi_\infty\} & (Q \nmid \mathfrak{n}) \end{cases}$$

gives an example of the set $I(\mathfrak{n}, Q)$ of Lemma 7.4.

Let $k, m \in \mathbb{Z}$. Since $\Gamma_0^\Theta(\mathfrak{n}) \subseteq SL_2(A)$, we have

$$M_{k,m}(\Gamma_0^\Theta(\mathfrak{n})) = M_k(\Gamma_0^\Theta(\mathfrak{n})), \quad S_{k,m}(\Gamma_0^\Theta(\mathfrak{n})) = S_k(\Gamma_0^\Theta(\mathfrak{n})).$$

Definition 7.6. Let $k \in \mathbb{Z}$ and let $Q \neq 0 \in A$ be a monic irreducible polynomial. Define

$$T_Q : M_k(\Gamma_0^\Theta(\mathfrak{n})) \to M_k(\Gamma_0^\Theta(\mathfrak{n})), \quad f \mapsto \sum f \Big|_k \Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n}).$$

We call it the Hecke operator at Q. When $Q \mid \mathfrak{n}$, we also write U_Q for T_Q.

By Lemma 7.4, the definition of T_Q is independent of Θ and thus for any subgroups $\Theta \subseteq \Theta'$ of $(A/(\mathfrak{n}))^\times$ we have commutative diagrams

$$\begin{tikzcd} M_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[r, "T_Q"] \arrow[d] & M_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[d] \\ M_k(\Gamma_0^{\Theta}(\mathfrak{n})) \arrow[r, "T_Q"'] & M_k(\Gamma_0^{\Theta}(\mathfrak{n})), \end{tikzcd} \qquad \begin{tikzcd} S_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[r, "T_Q"] \arrow[d] & S_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[d] \\ S_k(\Gamma_0^{\Theta}(\mathfrak{n})) \arrow[r, "T_Q"'] & S_k(\Gamma_0^{\Theta}(\mathfrak{n})), \end{tikzcd}$$

where the vertical arrows are natural inclusions.

Lemma 7.7. *Let $Q, Q' \in A \backslash \mathbb{F}_q$ be monic irreducible polynomials which are coprime to each other. Then*

$$T_Q \circ T_{Q'} = \left[\Gamma_0^{\Theta}(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & QQ' \end{pmatrix} \Gamma_0^{\Theta}(\mathfrak{n}) \right].$$

In particular, we have $T_Q \circ T_{Q'} = T_{Q'} \circ T_Q$.

Proof. For $\Gamma = \Gamma_0^{\Theta}(\mathfrak{n})$ and any nonzero $R \in A$, put

$$\Gamma_R = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}^{-1} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \cap \Gamma.$$

Write

$$\Gamma = \coprod_{i \in I} \Gamma_Q \eta_i, \quad \Gamma = \coprod_{j \in J} \Gamma_{Q'} \eta'_j.$$

By Example 7.5, we may assume $\eta_i, \eta'_j \in \Gamma_1(\mathfrak{n})$. Put

$$\eta_{i,Q'} := \begin{pmatrix} 1 & 0 \\ 0 & Q' \end{pmatrix}^{-1} \eta_i \begin{pmatrix} 1 & 0 \\ 0 & Q' \end{pmatrix}.$$

We claim that η_i can be chosen to satisfy $\eta_{i,Q'} \in \Gamma_1(\mathfrak{n})$, $\eta_{i,Q'} \equiv \mathrm{id} \bmod Q'$ and the set $\{\eta_{i,Q'} \mid i \in I\}$ forms a complete set of representatives of $\Gamma_Q \backslash \Gamma$. Indeed, when $Q \mid \mathfrak{n}$ the choice of Example 7.5 suffices. When $Q \nmid \mathfrak{n}$, we can find $R, S \in A$ satisfying $RQ - \mathfrak{n}(Q')^2 S = 1$ and replacing η_∞ in Example 7.5 by $\begin{pmatrix} RQ & S \\ \mathfrak{n}(Q')^2 & 1 \end{pmatrix}$ shows the claim.

With such a choice of η_i, we have

$$\bigcup_{i \in I,\ j \in J} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_i \begin{pmatrix} 1 & 0 \\ 0 & Q' \end{pmatrix} \eta'_j = \bigcup_{i \in I,\ j \in J} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & QQ' \end{pmatrix} \eta_{i,Q} \eta'_j.$$

Suppose that we have $\gamma \eta_{i_1,Q} \eta'_{j_1} = \eta_{i_2,Q} \eta'_{j_2}$ with some $\gamma \in \Gamma_{QQ'}$, $i_1, i_2 \in I$ and $j_1, j_2 \in J$. By Lemma 7.4, if $j_1 \neq j_2$ then the set

$$\{\eta_{i_1,Q} \eta'_{j_1}, \eta_{i_2,Q} \eta'_{j_2}, \eta'_j\ (j \neq j_1, j_2)\}$$

forms a complete set of representatives of $\Gamma_{Q'}\backslash\Gamma$. Since $\Gamma_{QQ'} \subseteq \Gamma_{Q'}$, this yields $\gamma = \mathrm{id}$, $j_1 = j_2$ and $i_1 = i_2$. By Lemma 7.1 these unions are disjoint, from which the lemma follows. $\square$

7.3. **Diamond operators.** Let $\mathfrak{n}$ be a nonzero element and let $\Theta \subseteq (A/(\mathfrak{n}))^\times$ be a subgroup. For any $d \in A$ which is coprime to $\mathfrak{n}$, we can find $\eta_d \in \Gamma_0(\mathfrak{n})$ satisfying $\eta_d = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we have

$$\eta_d^{-1}\Gamma_0^\Theta(\mathfrak{n})\eta_d = \Gamma_0^\Theta(\mathfrak{n}). \tag{7.1}$$

Lemma 7.8. *Let $d \in A$ be an element which is coprime to $\mathfrak{n}$. Then $f \mapsto f|_k\eta_d$ defines endomorphisms*

$$\langle d\rangle_\mathfrak{n} : M_k(\Gamma_0^\Theta(\mathfrak{n})) \to M_k(\Gamma_0^\Theta(\mathfrak{n})), \quad \langle d\rangle_\mathfrak{n} : S_k(\Gamma_0^\Theta(\mathfrak{n})) \to S_k(\Gamma_0^\Theta(\mathfrak{n})),$$

which depend only on $d \bmod \mathfrak{n}$ and satisfy

$$\langle d\rangle_\mathfrak{n} \circ \langle d'\rangle_\mathfrak{n} = \langle dd'\rangle_\mathfrak{n} = \langle d'\rangle_\mathfrak{n} \circ \langle d\rangle_\mathfrak{n}.$$

Proof. Since the map

$$\Gamma_0(\mathfrak{n}) \to (A/(\mathfrak{n}))^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod \mathfrak{n}$$

is a group homomorphism with kernel $\Gamma_1(\mathfrak{n})$, if d and d' are elements of A which are coprime to $\mathfrak{n}$ and satisfy $d \equiv d' \bmod \mathfrak{n}$, then with any choices of η_d and $\eta_{d'}$ we have $\eta_{d'} \in \Gamma_1(\mathfrak{n})\eta_d \subseteq \Gamma_0^\Theta(\mathfrak{n})\eta_d$. Thus the lemma follows from Lemma 6.62 and (7.1). $\square$

Definition 7.9. The operator $\langle d\rangle_\mathfrak{n}$ is called the diamond operator of level $\Gamma_0^\Theta(\mathfrak{n})$.

Since the definition of $\langle d\rangle_\mathfrak{n}$ is independent of Θ, for any subgroups $\Theta \subseteq \Theta'$ of $(A/(\mathfrak{n}))^\times$ we have commutative diagrams

$$\begin{tikzcd} M_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[r, "\langle d\rangle_\mathfrak{n}"] \arrow[d] & M_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[d] & S_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[r, "\langle d\rangle_\mathfrak{n}"] \arrow[d] & S_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[d] \\ M_k(\Gamma_0^{\Theta}(\mathfrak{n})) \arrow[r, "\langle d\rangle_\mathfrak{n}"'] & M_k(\Gamma_0^{\Theta}(\mathfrak{n})), & S_k(\Gamma_0^{\Theta}(\mathfrak{n})) \arrow[r, "\langle d\rangle_\mathfrak{n}"'] & S_k(\Gamma_0^{\Theta}(\mathfrak{n})), \end{tikzcd}$$

where the vertical arrows are natural inclusions.

Lemma 7.10. *Let Θ be a subgroup of $(A/(\mathfrak{n}))^\times$. Then $f \in M_k(\Gamma_1(\mathfrak{n}))$ lies in $M_k(\Gamma_0^\Theta(\mathfrak{n}))$ if and only if $\langle d\rangle_\mathfrak{n} f = f$ for any $d \in A$ satisfying $d \bmod \mathfrak{n} \in \Theta$.*

Proof. The group $\Gamma_0^\Theta(\mathfrak{n})$ is generated by its subgroup $\Gamma_1(\mathfrak{n})$ and

$$\{\eta_d \mid d \in A,\ d \bmod \mathfrak{n} \in \Theta\},$$

from which the equivalence follows. □

Lemma 7.11. *For any monic irreducible polynomial $Q \in A\backslash\mathbb{F}_q$ and any $d \in A$ coprime to $\mathfrak{n}$, we have*

$$\Gamma_0^\Theta(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\Gamma_0^\Theta(\mathfrak{n}) = \Gamma_0^\Theta(\mathfrak{n})\eta_d^{-1}\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\eta_d\Gamma_0^\Theta(\mathfrak{n}).$$

Proof. Write $\eta_d = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0(\mathfrak{n})$. Then

$$\eta_d^{-1}\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\eta_d = \begin{pmatrix}ad-bcQ & bd(1-Q)\\ ac(Q-1) & adQ-bc\end{pmatrix}.$$

Suppose $Q \nmid a$. We can find $\beta \in A$ satisfying $\deg(\beta) < \deg(Q)$ and $a\beta \equiv b \bmod Q$. Then we have

$$\begin{aligned}
&\begin{pmatrix}ad-bcQ & bd(1-Q)\\ ac(Q-1) & adQ-bc\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & Q\end{pmatrix}^{-1}\\
&\quad= \frac{1}{Q}\begin{pmatrix}ad-bcQ & bd(1-Q)\\ ac(Q-1) & adQ-bc\end{pmatrix}\begin{pmatrix}Q & -\beta\\ 0 & 1\end{pmatrix}\\
&\quad= \begin{pmatrix}ad-bcQ & bc\beta-bd+dQ^{-1}(b-a\beta)\\ ac(Q-1) & -ac\beta+ad+cQ^{-1}(a\beta-b)\end{pmatrix},
\end{aligned}$$

which lies in $\Gamma_1(\mathfrak{n})$ since $ad-bc=1$ and $\mathfrak{n} \mid c$.

Suppose $Q \mid a$. Since $ad-bc=1$, it forces $Q \nmid \mathfrak{n}$. We can find $R, S \in A$ satisfying $RQ - \mathfrak{n}S = 1$. Then we have

$$\begin{aligned}
&\begin{pmatrix}ad-bcQ & bd(1-Q)\\ ac(Q-1) & adQ-bc\end{pmatrix}\begin{pmatrix}RQ & S\\ \mathfrak{n}Q & Q\end{pmatrix}^{-1}\\
&\quad= \frac{1}{Q}\begin{pmatrix}ad-bcQ & bd(1-Q)\\ ac(Q-1) & adQ-bc\end{pmatrix}\begin{pmatrix}Q & -S\\ -\mathfrak{n}Q & RQ\end{pmatrix}\\
&\quad= \begin{pmatrix}ad-bcQ-\mathfrak{n}bd(1-Q) & -S(Q^{-1}ad-bc)+bdR(1-Q)\\ ac(Q-1)-\mathfrak{n}(adQ-bc) & -SQ^{-1}ac(Q-1)+R(adQ-bc)\end{pmatrix},
\end{aligned}$$

which again lies in $\Gamma_1(\mathfrak{n})$.

By Example 7.5, this implies

$$\Gamma_0^\Theta(\mathfrak{n})\eta_d^{-1}\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\eta_d\Gamma_0^\Theta(\mathfrak{n}) \subseteq \Gamma_0^\Theta(\mathfrak{n})\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}\Gamma_0^\Theta(\mathfrak{n}).$$

Since $\eta_d^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, the containment above for η_a gives

$$\Gamma_0^\Theta(\mathfrak{n})\eta_d \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_d^{-1}\Gamma_0^\Theta(\mathfrak{n}) \subseteq \Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n}).$$

By (7.1), this yields

$$\Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n}) \subseteq \Gamma_0^\Theta(\mathfrak{n})\eta_d^{-1} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_d\Gamma_0^\Theta(\mathfrak{n})$$

and the lemma follows. □

Lemma 7.12. *For any monic irreducible polynomial $Q \in A\backslash\mathbb{F}_q$ and any $d \in A$ coprime to $\mathfrak{n}$, we have*

$$T_Q \circ \langle d\rangle_{\mathfrak{n}} = \langle d\rangle_{\mathfrak{n}} \circ T_Q \quad on \quad M_k(\Gamma_0^\Theta(\mathfrak{n})).$$

Proof. By Lemma 7.11 and (7.1), we have

$$\begin{aligned}
\Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n})\eta_d &= \Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_d\Gamma_0^\Theta(\mathfrak{n}) \\
&= \Gamma_0^\Theta(\mathfrak{n})\eta_d\eta_d^{-1} \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \eta_d\Gamma_0^\Theta(\mathfrak{n}) \\
&= \eta_d\Gamma_0^\Theta(\mathfrak{n}) \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n}).
\end{aligned}$$

Combining this with (7.1), we obtain

$$\coprod_{\xi \in I(\mathfrak{n},Q)} \Gamma_0^\Theta(\mathfrak{n})\xi\eta_d = \coprod_{\xi \in I(\mathfrak{n},Q)} \eta_d\Gamma_0^\Theta(\mathfrak{n})\xi = \coprod_{\xi \in I(\mathfrak{n},Q)} \Gamma_0^\Theta(\mathfrak{n})\eta_d\xi,$$

from which the lemma follows. □

Remark 7.13. For any character $\chi : (A/(\mathfrak{n}))^\times \to \mathbb{C}_\infty^\times$, we denote by $M_k(\Gamma_0(\mathfrak{n}), \chi)$ the subspace of $M_k(\Gamma_1(\mathfrak{n}))$ on which $\langle d\rangle_{\mathfrak{n}} = \chi(d)\mathrm{id}$ for any $d \in (A/(\mathfrak{n}))^\times$. Contrary to the case of elliptic modular forms, the $\mathbb{C}_\infty$-vector space $M_k(\Gamma_1(\mathfrak{n}))$ is not necessarily the direct sum of $M_k(\Gamma_0(\mathfrak{n}), \chi)$. This is because the order of $(A/(\mathfrak{n}))^\times$ may be divisible by $p = \mathrm{char}(\mathbb{C}_\infty)$ and a representation of $(A/(\mathfrak{n}))^\times$ over $\mathbb{C}_\infty$ is not necessarily semi-simple. Instead, for any subgroup $\Theta \subseteq (A/(\mathfrak{n}))^\times$ of index prime to p, we do have a decomposition

$$M_k(\Gamma_1(\mathfrak{n})) = \bigoplus_{\chi} M_k(\Gamma_0^\Theta(\mathfrak{n}), \chi),$$

where the sum runs over the set of characters $(A/(\mathfrak{n}))^\times/\Theta \to \mathbb{C}_\infty^\times$.

7.4. **Type operators.**

Definition 7.14. For any nonzero monic polynomial $\mathfrak{n} \in A$, let

$$G\Gamma_0^\Theta(\mathfrak{n}) = \begin{pmatrix} 1 & 0 \\ 0 & \mathbb{F}_q^\times \end{pmatrix} \Gamma_0^\Theta(\mathfrak{n}).$$

When $\Theta = (A/(\mathfrak{n}))^\times$ or $\{1\}$, we write it as $G\Gamma_0(\mathfrak{n})$ or $G\Gamma_1(\mathfrak{n})$. When $\mathfrak{n} = 1$, we have $G\Gamma_0(\mathfrak{n}) = G\Gamma_1(\mathfrak{n}) = GL_2(A)$.

Definition 7.15. For any $\lambda \in \mathbb{F}_q^\times$, the element $\tau_\lambda = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ satisfies

$$\tau_\lambda^{-1} \Gamma_0^\Theta(\mathfrak{n}) \tau_\lambda = \Gamma_0^\Theta(\mathfrak{n}). \tag{7.2}$$

By Lemma 6.62, the map $f \mapsto f|_k \tau_\lambda$ give endomorphisms

$$\{\lambda\} : M_k(\Gamma_0^\Theta(\mathfrak{n})) \to M_k(\Gamma_0^\Theta(\mathfrak{n})), \quad \{\lambda\} : S_k(\Gamma_0^\Theta(\mathfrak{n})) \to S_k(\Gamma_0^\Theta(\mathfrak{n})),$$

which we call the type operators.

Since the definition of $\{\lambda\}$ is independent of Θ, for any subgroups $\Theta \subseteq \Theta'$ of $(A/(\mathfrak{n}))^\times$ we have commutative diagrams

$$\begin{tikzcd} M_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[r, "\{\lambda\}"] \arrow[d] & M_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[d] \\ M_k(\Gamma_0^{\Theta}(\mathfrak{n})) \arrow[r, "\{\lambda\}"'] & M_k(\Gamma_0^{\Theta}(\mathfrak{n})), \end{tikzcd} \qquad \begin{tikzcd} S_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[r, "\{\lambda\}"] \arrow[d] & S_k(\Gamma_0^{\Theta'}(\mathfrak{n})) \arrow[d] \\ S_k(\Gamma_0^{\Theta}(\mathfrak{n})) \arrow[r, "\{\lambda\}"'] & S_k(\Gamma_0^{\Theta}(\mathfrak{n})), \end{tikzcd}$$

where the vertical arrows are natural inclusions.

For any integer n, we denote by

$$M_k(\Gamma_0^\Theta(\mathfrak{n}))\{n\}, \quad S_k(\Gamma_0^\Theta(\mathfrak{n}))\{n\}$$

the subspaces of $M_k(\Gamma_0^\Theta(\mathfrak{n}))$ and $S_k(\Gamma_0^\Theta(\mathfrak{n}))$ on which $\{\lambda\}$ acts by λ^n for any $\lambda \in \mathbb{F}_q^\times$, respectively.

Lemma 7.16. *For any integer m, we have*

$$M_{k,m}(G\Gamma_0^\Theta(\mathfrak{n})) = M_k(\Gamma_0^\Theta(\mathfrak{n}))\{k-m-1\}, \quad S_{k,m}(G\Gamma_0^\Theta(\mathfrak{n})) = S_k(\Gamma_0^\Theta(\mathfrak{n}))\{k-m-1\}.$$

Proof. For any $f \in M_k(\Gamma_0^\Theta(\mathfrak{n}))$, we have

$$(\{\lambda\}f)(z) = \lambda^{-1} f(\lambda^{-1} z).$$

On the other hand, $M_{k,m}(G\Gamma_0^\Theta(\mathfrak{n}))$ is the subspace of $M_k(\Gamma_0^\Theta(\mathfrak{n}))$ consisting of f satisfying $f|_{k,m}\tau_\lambda = f$ for any $\lambda \in \mathbb{F}_q^\times$. Then the lemma follows from

$$(f|_{k,m}\tau_\lambda)(z) = \lambda^{m-k} f(\lambda^{-1} z) = \lambda^{m-k+1}(\{\lambda\}f)(z).$$

□

Lemma 7.17. *For any monic irreducible polynomial $Q \in A\backslash\mathbb{F}_q$, any $d \in A$ coprime to $\mathfrak{n}$ and any $\lambda \in \mathbb{F}_q^\times$, we have*

$$T_Q \circ \{\lambda\} = \{\lambda\} \circ T_Q, \quad \langle d\rangle_{\mathfrak{n}} \circ \{\lambda\} = \{\lambda\} \circ \langle d\rangle_{\mathfrak{n}}.$$

In particular, for any $m \in \mathbb{Z}$, the operators T_Q and $\langle d\rangle_{\mathfrak{n}}$ for $M_k(\Gamma_0^\Theta(\mathfrak{n}))$ induce endomorphisms

$$T_Q : M_{k,m}(G\Gamma_0^\Theta(\mathfrak{n})) \to M_{k,m}(G\Gamma_0^\Theta(\mathfrak{n})),$$
$$\langle d\rangle_{\mathfrak{n}} : M_{k,m}(G\Gamma_0^\Theta(\mathfrak{n})) \to M_{k,m}(G\Gamma_0^\Theta(\mathfrak{n}))$$

which stabilize $S_{k,m}(G\Gamma_0^\Theta(\mathfrak{n}))$.

Proof. By (7.2), we have

$$\Gamma_0^\Theta(\mathfrak{n})\begin{pmatrix}1&0\\0&Q\end{pmatrix}\Gamma_0^\Theta(\mathfrak{n})\tau_\lambda = \Gamma_0^\Theta(\mathfrak{n})\begin{pmatrix}1&0\\0&Q\end{pmatrix}\tau_\lambda\Gamma_0^\Theta(\mathfrak{n})$$
$$= \Gamma_0^\Theta(\mathfrak{n})\tau_\lambda\begin{pmatrix}1&0\\0&Q\end{pmatrix}\Gamma_0^\Theta(\mathfrak{n}) = \tau_\lambda\Gamma_0^\Theta(\mathfrak{n})\begin{pmatrix}1&0\\0&Q\end{pmatrix}\Gamma_0^\Theta(\mathfrak{n}).$$

This and (7.2) give

$$\coprod_{\xi\in I(\mathfrak{n},Q)} \Gamma_0^\Theta(\mathfrak{n})\xi\tau_\lambda = \coprod_{\xi\in I(\mathfrak{n},Q)} \Gamma_0^\Theta(\mathfrak{n})\tau_\lambda\xi,$$

which yields the first equality of the lemma. For the second, the element $\eta_d = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma_0(\mathfrak{n})$ satisfies

$$\tau_\lambda^{-1}\eta_d\tau_\lambda = \begin{pmatrix}a&b\lambda\\c\lambda^{-1}&d\end{pmatrix} \in \Gamma_0(\mathfrak{n}).$$

Thus the element $\tau_\lambda^{-1}\eta_d\tau_\lambda$ also acts on $M_k(\Gamma_0^\Theta(\mathfrak{n}))$ as $\langle d\rangle_{\mathfrak{n}}$ and we obtain the second equality. The last assertion follows from Lemma 7.16. $\square$

Example 7.18. Let $Q \in A\backslash\mathbb{F}_q$ be a monic irreducible polynomial. Then the element $\eta_Q := \begin{pmatrix}R&S\\\mathfrak{n}&Q\end{pmatrix}$ in Example 7.5 acts on $M_k(\Gamma_0^\Theta(\mathfrak{n}))$ as the diamond operator $\langle Q\rangle_{\mathfrak{n}}$. Thus, for any $f \in M_k(\Gamma_0^\Theta(\mathfrak{n}))$ we have

$$(T_Qf)(z) = \begin{cases} \frac{1}{Q}\sum_{\deg(\beta)<\deg(Q)} f\left(\frac{z+\beta}{Q}\right) + Q^{k-1}(\langle Q\rangle_{\mathfrak{n}}f)(Qz) & (Q \nmid \mathfrak{n}),\\ \frac{1}{Q}\sum_{\deg(\beta)<\deg(Q)} f\left(\frac{z+\beta}{Q}\right) & (Q \mid \mathfrak{n}).\end{cases}$$

For $\Theta = (A/(\mathfrak{n}))^\times$, by Lemma 7.10 the action of $\langle Q\rangle_{\mathfrak{n}}$ on $M_k(\Gamma_0(\mathfrak{n}))$ is trivial. Hence, for any $f \in M_k(\Gamma_0(\mathfrak{n}))$ we have

$$(T_Qf)(z) = \begin{cases} \frac{1}{Q}\sum_{\deg(\beta)<\deg(Q)} f\left(\frac{z+\beta}{Q}\right) + Q^{k-1}f(Qz) & (Q \nmid \mathfrak{n}),\\ \frac{1}{Q}\sum_{\deg(\beta)<\deg(Q)} f\left(\frac{z+\beta}{Q}\right) & (Q \mid \mathfrak{n}).\end{cases}$$

Remark 7.19. In the literature, there are different normalizations of Hecke operators. We adopt the one in [Böc, Example 6.13] which is parallel to the case of elliptic modular forms as [DS, Proposition 5.2.1]. On the other hand, as [Gos1, Remark 3.6] and [Gek2, (7.1)], the operator QT_Q in our notation is sometimes called the Hecke operator at Q.

7.5. **Hecke operators for non-irreducible polynomials.** As in the classical case [Miy, §4.5], when the level is $\Gamma_0(\mathfrak{n})$ there is a standard way to define Hecke operators at Q even when Q is not irreducible. For this, let $\mathfrak{n} \in A$ be a nonzero element and

$$\Delta_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) \;\middle|\; c \equiv 0 \bmod \mathfrak{n},\ (a, \mathfrak{n}) = (1),\ ad - bc \in A_+ \right\}.$$

Lemma 7.20 ([Miy], Lemma 4.5.2)**.** *For any $\xi \in \Delta_0(\mathfrak{n})$, there exist unique $Q_1, Q_2 \in A_+$ such that $Q_1 \mid Q_2$, $(Q_1, \mathfrak{n}) = (1)$ and*

$$\Gamma_0(\mathfrak{n})\xi\Gamma_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n}) \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \Gamma_0(\mathfrak{n}).$$

Proof. Put

$$L = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \;\middle|\; u, v \in A \right\}, \quad L_0 = \left\{ \begin{pmatrix} u \\ \mathfrak{n}v \end{pmatrix} \;\middle|\; u, v \in A \right\},$$

on which $M_2(A)$ acts via the left multiplication. Note that we have $\xi L_0 \subseteq L_0$.

For any free A-submodules $L_1 \supseteq L_2$ of rank two of A^2, write

$$[L_1 : L_2] := \mathrm{Ann}_A \left(\bigwedge_A^2 L_1 / \bigwedge_A^2 L_2 \right).$$

For any $L_1 \supseteq L_2 \supseteq L_3$, we have $[L_1 : L_3] = [L_1 : L_2][L_2 : L_3]$.

Let $D = \det(\xi)$. Then

$$[L : \xi L_0] = [L : \xi L][\xi L : \xi L_0] = (D\mathfrak{n}).$$

Thus we can find a basis w_1, w_2 of the A-module L satisfying

$$\xi L_0 = A(aw_1) \oplus A(bw_2), \quad a \mid b,\ ab = D\mathfrak{n}$$

with some $a, b \in A_+$.

Since the $(1,1)$-entry of ξ is coprime to $\mathfrak{n}$, we have $\xi L_0 \nsubseteq tL$ for any non-constant divisor t of $\mathfrak{n}$. This yields $(a, \mathfrak{n}) = (1)$ and $\mathfrak{n} \mid b$. Since the A-module L/L_0 is isomorphic to $A/(\mathfrak{n})$, the image of L_0 by the natural map

$$L \to L/\xi L_0 \simeq A/(a)w_1 \oplus A/(b)w_2 \tag{7.3}$$

equals $A/(a)w_1 \oplus (\mathfrak{n})/(b)w_2$. Thus we obtain

$$L_0 = Aw_1 \oplus A(\mathfrak{n}w_2).$$

Put

$$L' = A(aw_1) \oplus A(\frac{b}{\mathfrak{n}}w_2),$$

so that $\xi L_0 \subseteq L'$. Since the image of ξL by the map (7.3) is killed by $\mathfrak{n}$ and $(a, \mathfrak{n}) = (1)$, we have $\xi L \subseteq L'$. By the equalities $[\xi L : \xi L_0] = [L : L_0] = q^{\deg(\mathfrak{n})} = [L' : \xi L_0]$, we obtain $\xi L = L'$.

Now we define $\gamma_1, \gamma_2 \in GL_2(A)$ by

$$(w_1, w_2) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)\gamma_1, \quad \left(\xi\begin{pmatrix}1\\0\end{pmatrix}, \xi\begin{pmatrix}0\\1\end{pmatrix}\right) = \left(aw_1, \frac{b}{\mathfrak{n}}w_2\right)\gamma_2,$$

so that

$$\xi = \gamma_1 \begin{pmatrix} a & 0 \\ 0 & \frac{b}{\mathfrak{n}} \end{pmatrix} \gamma_2.$$

By replacing w_i with its multiple by $\mathbb{F}_q^\times$, we may assume $\det(\gamma_1) = 1$. Since $\det(\xi)$, $\mathfrak{n}$, a, b are all monic, we also have $\det(\gamma_2) = 1$. Since $w_1 \in L_0$ and $\xi \begin{pmatrix}1\\0\end{pmatrix} \in \xi L_0$, we have $\gamma_i \in \Gamma_0(\mathfrak{n})$ for $i = 1, 2$. Hence we obtain the equality of the lemma with $(Q_1, Q_2) = (a, \frac{b}{\mathfrak{n}})$.

If we have two pairs (Q_1, Q_2) and (Q_1', Q_2') as in the lemma, then we have

$$GL_2(A)\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} GL_2(A) = GL_2(A)\begin{pmatrix} Q_1' & 0 \\ 0 & Q_2' \end{pmatrix} GL_2(A).$$

By the theory of Smith normal forms, it forces $Q_1 = Q_1'$ and $Q_2 = Q_2'$. This concludes the proof of the lemma. □

Definition 7.21. For any monic polynomials $Q, Q_1, Q_2 \in A$ satisfying $Q_1 \mid Q_2$ and $(Q_1, \mathfrak{n}) = (1)$, we define

$$T(Q_1, Q_2) = \left[\Gamma_0(\mathfrak{n})\begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}\Gamma_0(\mathfrak{n})\right],$$
$$T(Q) = \sum_{Q_1 Q_2 = Q} T(Q_1, Q_2),$$

where the sum runs over the set of pairs (Q_1, Q_2) satisfying

$$Q_1, Q_2 \in A_+, \quad Q_1 Q_2 = Q, \quad Q_1 \mid Q_2, \quad (Q_1, \mathfrak{n}) = (1).$$

When Q is irreducible, we see that $T(Q)$ agrees with T_Q as an endomorphism of $M_k(\Gamma_0(\mathfrak{n}))$.

Remark 7.22. As in the proof of [Miy, Lemma 4.5.7], we can show

$$T(Q)T(Q^e) = T(Q^{e+1})$$

for any Q, since our Hecke operators act on vector spaces over $\mathbb{C}_\infty$ which has characteristic p. That is, Hecke operators acting on $M_k(\Gamma_0(\mathfrak{n}))$ are multiplicative. This indicates that the Galois representation attached to a Drinfeld eigenform is one-dimensional, as Böckle proved in [Böc].

8. Examples of Drinfeld modular forms

8.1. **Goss polynomials.**

Lemma 8.1. *Let $n \geqslant 1$ be an integer and let $t_1, \ldots, t_n$ be indeterminates. Write*

$$f(X) := \prod_{i=1}^{n}(X - t_i) = \sum_{i=0}^{n} a_i X^{n-i}, \quad a_i \in \mathbb{Z}[t_1, \ldots, t_n].$$

For any integer $k \geqslant 1$, let $S_k = \sum_{i=1}^{n} t_i^k$. Then we have

$$\begin{aligned} S_k + a_1 S_{k-1} + \cdots + a_{k-1} S_1 + k a_k &= 0 \quad (k \leqslant n), \\ S_k + a_1 S_{k-1} + \cdots + a_{n-1} S_{k-n+1} + a_n S_{k-n} &= 0 \quad (k > n). \end{aligned}$$

Proof. We have

$$f'(X) = \sum_{i=0}^{n}(n-i) a_i X^{n-i-1}.$$

On the other hand, if we embed $\mathbb{Q}(t_1, \ldots, t_n)(X)$ into $\mathbb{Q}(t_1, \ldots, t_n)((1/X))$ naturally, we have

$$\frac{f'(X)}{f(X)} = \sum_{i=1}^{n} \frac{1}{X - t_i} = \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{t_i^k}{X^{k+1}} = \sum_{k=0}^{\infty} \frac{S_k}{X^{k+1}}.$$

Multiplying $f(X)$, we obtain

$$f'(X) = \sum_{i=0}^{n} a_i X^{n-i} \sum_{k=0}^{\infty} \frac{S_k}{X^{k+1}} = \sum_{k=0}^{\infty} \sum_{i=0}^{n} a_i S_k X^{n-i-k-1}.$$

Comparing two expressions of $f'(X)$ yields the lemma. $\square$

For any $\mathbb{F}_q$-subspace Λ of $\mathbb{C}_\infty$ of finite dimension, put $m = \dim_{\mathbb{F}_q}(\Lambda)$ and

$$\begin{aligned} e_\Lambda(X) &= X \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{X}{\lambda}\right) = \sum_{i=0}^{m} \alpha_i X^{q^i} \in \mathbb{C}_\infty[X], \\ U_\Lambda(X) &= \frac{1}{e_\Lambda(X)} \in \mathbb{C}_\infty(X), \\ S_{\Lambda,k}(X) &= \sum_{\lambda \in \Lambda} \frac{1}{(X+\lambda)^k} \in \mathbb{C}_\infty(X). \end{aligned}$$

Then $\alpha_0 = 1$.

Lemma 8.2. *Let Λ be any $\mathbb{F}_q$-subspace of $\mathbb{C}_\infty$ of finite dimension and let $k \geqslant 1$ be any integer. Then there exists a monic polynomial $G_{\Lambda,k}(X) \in \mathbb{C}_\infty[X]$ of degree k with $G_{\Lambda,k}(0) = 0$ satisfying*

$$S_{\Lambda,k}(X) = G_{\Lambda,k}(U_\Lambda(X)).$$

Moreover, for $k \geqslant 2$ we have

$$G_{\Lambda,k}(X) = X \sum_{0 \leqslant i \leqslant \lfloor \log_q(k) \rfloor} \alpha_i G_{\Lambda,k-q^i}(X). \tag{8.1}$$

Proof. For $k = 1$, we have $e'_\Lambda(X) = 1$ and

$$\frac{1}{e_\Lambda(X)} = \frac{e'_\Lambda(X)}{e_\Lambda(X)} = \sum_{\lambda \in \Lambda} \frac{1}{X+\lambda}.$$

Hence the polynomial $G_{\Lambda,1}(X) = X$ satisfies the condition.

Let $k \geqslant 2$. Put $m = \dim_{\mathbb{F}_q}(\Lambda)$. For an indeterminate Z, consider the polynomial

$$f(X) = e_\Lambda(X - Z) = (X - Z) \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{X - Z}{\lambda}\right) \in \mathbb{C}_\infty(Z)[X],$$

which we can write as

$$f(X) = e_\Lambda(X) - e_\Lambda(Z) = \sum_{i=0}^{m} \alpha_i X^{q^i} - e_\Lambda(Z).$$

Then we have $\deg(f(X)) = q^m$ and the set of roots of $f(X)$ is $\{Z + \lambda \mid \lambda \in \Lambda\}$.

We denote the reciprocal polynomial of $f(X)$ by

$$\tilde{f}(X) := X^{q^m} f(X^{-1}) = \sum_{i=0}^{m} \alpha_i X^{q^m - q^i} - e_\Lambda(Z) X^{q^m} \in \mathbb{C}_\infty(Z)[X],$$

whose set of roots is $\{(Z+\lambda)^{-1} \mid \lambda \in \Lambda\}$. For $k \leqslant q^m$, the coefficient of X^{q^m-k} in $\tilde{f}(X)$ is zero unless $k = q^i$ with some $i > 0$, and thus the term ka_k in Lemma 8.1 vanishes. Now Lemma 8.1 implies

$$S_{\Lambda,k}(Z) + \frac{\alpha_0}{-e_\Lambda(Z)} S_{\Lambda,k-1}(Z) + \frac{\alpha_1}{-e_\Lambda(Z)} S_{\Lambda,k-q}(Z) + \cdots = 0,$$

which yields

$$S_{\Lambda,k}(Z) = U_\Lambda(Z)(S_{\Lambda,k-1}(Z) + \alpha_1 S_{\Lambda,k-q}(Z) + \alpha_2 S_{\Lambda,k-q^2}(Z) + \cdots). \tag{8.2}$$

Hence the polynomial $G_{\Lambda,k}(X)$ defined inductively by (8.1) satisfies the condition of the lemma. □

Let $\Lambda \subseteq \mathbb{C}_\infty$ be an $\mathbb{F}_q$-lattice. For any positive rational number ρ, the subset $\Lambda^{\leqslant \rho}$ is an $\mathbb{F}_q$-subspace of $\mathbb{C}_\infty$ of finite dimension and $S_{\Lambda^{\leqslant \rho},k}(X)$ is a rigid analytic function on Ω for any integer $k \geqslant 1$.

Lemma 8.3. *Let Λ be any $\mathbb{F}_q$-lattice of $\mathbb{C}_\infty$ and let $k \geqslant 1$ an integer. For any non-negative integers r, s, the sequence*

$$\{S_{\Lambda^{\leqslant q^n},k}(X)\}_{n \in \mathbb{Z}_{\geqslant 0}}$$

converges in $\mathcal{O}(\Omega_{r,s})$ and its limit $S_{\Lambda,k}(X)$ is a rigid analytic function on Ω.

Proof. Since $\Omega_{r,s}$ is a reduced affinoid variety, it is enough to show the convergence with respect to the supremum norm. For any integer $n \geqslant s$, $\lambda \in \Lambda \backslash \Lambda^{\leqslant q^n}$ and $z \in \Omega_{r,s}$, we have

$$|z + \lambda| = |\lambda| > q^n$$

and thus for any integers $m \geqslant n \geqslant s$ we obtain

$$|S_{\Lambda^{\leqslant q^n},k}(X) - S_{\Lambda^{\leqslant q^m},k}(X)|_{\sup} < q^{-nk}$$

on $\Omega_{r,s}$. Thus the sequence of the lemma is Cauchy. The last assertion follows from the continuity of restriction maps. □

Proposition 8.4 ([Gek2], Proposition (3.4))**.** *Let Λ be any $\mathbb{F}_q$-lattice of $\mathbb{C}_\infty$ and let $k \geqslant 1$ be any integer. Write*

$$\exp_\Lambda(X) = \sum_{i=0}^{\infty} \alpha_i Z^{q^i}, \quad \alpha_i \in \mathbb{C}_\infty.$$

Then there exists a monic polynomial $G_{\Lambda,k}(X) \in \mathbb{C}_\infty[X]$ of degree k with $G_{\Lambda,k}(0) = 0$ satisfying

$$S_{\Lambda,k}(X) = G_{\Lambda,k}(\bar{\pi} u_\Lambda(X)) = G_{\Lambda,k}\left(\frac{1}{\exp_\Lambda(X)}\right).$$

Moreover, for $k \geqslant 2$ we have

$$G_{\Lambda,k}(X) = X \sum_{0 \leqslant i \leqslant \lfloor \log_q(k) \rfloor} \alpha_i G_{\Lambda,k-q^i}(X). \tag{8.3}$$

Proof. Since $S_{\Lambda^{\leqslant q^n},1}(X) = U_{\Lambda^{\leqslant q^n}}(X)$, taking the limit we see that $G_{\Lambda,1}(X) = X$ satisfies the condition.

Suppose $k \geqslant 2$. Note that the convergence

$$\exp_\Lambda(X) = \lim_{n\to\infty} \exp_{\Lambda^{\leqslant q^n}}(X)$$

of Lemma 6.7 with respect to the ρ-Gauss norm implies that each coefficient of $\exp_{\Lambda^{\leqslant q^n}}(X)$ also converges to that of $\exp_\Lambda(X)$. Taking the limit of (8.2) for $\Lambda^{\leqslant q^n}$ shows that the same equality holds for Λ in $\mathcal{O}(\Omega_{r,s})$ and also in $\mathcal{O}(\Omega)$. Hence the polynomial $G_{\Lambda,k}(X)$ defined inductively by (8.3) satisfies the condition of the proposition. $\square$

Definition 8.5. We call $G_{\Lambda,k}(X)$ the k-th Goss polynomial for Λ.

8.2. Eisenstein series for $GL_2(A)$.

Lemma 8.6. *Let $w \geqslant 0$ be an integer. Let $P(X) \neq 0 \in \mathbb{C}_\infty[X]$ be a polynomial satisfying $P(0) = 0$. Suppose that for any $z \in \Omega$, the series*

$$f(z) = \sum_{a \in A_+} a^w P(u_A(az))$$

converges in $\mathbb{C}_\infty$. Then for any sufficiently small integer r, the series

$$F(u) = \sum_{a \in A_+} a^w P(f_a(u))$$

converges in $\mathcal{O}(D(0,\rho_r))$ with $\rho_r = |\bar{\pi}|^{-1}\sigma_{A,q^{-r}}^{-1}$ and satisfies

$$F(u_A(z)) = f(z) \quad \textit{for any } z \in \Omega_r.$$

Proof. By Lemma 6.12, the function $r \mapsto \rho_r$ is increasing. Since $\sigma_{A,q^{-r}} \geqslant q^{-r}$, we have $\lim_{r\to-\infty} \rho_r = 0$.

Write

$$P(X) = p_l X^l + p_{l+1}X^{l+1} \cdots + p_d X^d, \quad p_i \in \mathbb{C}_\infty,\ 0 < l \leqslant d$$

satisfying $p_l \neq 0$ and $p_d \neq 0$. Take any integer $r \leqslant N_A$ such that $\rho_r < q^{-1}$ and $|p_l| > |p_i|\rho_r$ for any $i > l$. Then, Lemma 6.40 implies that for any $j \in [1, d-l]$, any $a \in A_+$ with $\deg(a) = m$ and any $u \in D_{\mathbb{C}_\infty}(0,\rho_r)$, we have

$$|p_{j+l} f_a(u)^{j+l}| = |p_{j+l}||u|^{(j+l)q^m} \leqslant |p_{j+l}||u|^{1+lq^m} < |p_l||u|^{lq^m} = |p_l f_a(u)^l|$$

and thus $|P(f_a(u))| = |p_l||u|^{lq^m}$.

For the supremum norm on $D_{\mathbb{C}_\infty}(0,\rho_r)$, this yields

$$|a^w P(f_a(u))|_{\sup} \leqslant q^{mw}|p_l|\rho_r^{lq^m} < |p_l|q^{mw-lq^m}.$$

Since $l > 0$, we have $\lim_{m\to\infty} |a^w P(f_a(u))|_{\sup} = 0$ and the series $F(u)$ converges to define an element of $\mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho_r))$.

By (6.10), for any $z \in \Omega_r$ we have $u_A(z) \in D_{\mathbb{C}_\infty}(0,\rho_r)$. Hence Lemma 6.39 yields

$$F(u_A(z)) = \sum_{a\in A_+} a^w P(f_a(u_A(z))) = \sum_{a\in A_+} a^w P(u_A(az)) = f(z).$$

This concludes the proof. □

Lemma 8.7. *Let $\rho \in q^{\mathbb{Q}}$ and let $f \in \mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho)\backslash\{0\})$. Suppose $f(z) = 0$ for any $z \in D_{\mathbb{C}_\infty}(0,\rho)\backslash\{0\}$. Then $f = 0$.*

Proof. For any $\sigma \in q^{\mathbb{Q}}$ satisfying $\sigma \in (0,\rho]$, put

$$A_{\mathbb{C}_\infty}[\sigma,\rho] = \{z \in \mathbb{C}_\infty \mid \sigma \leqslant |z| \leqslant \rho\},$$

which is an admissible affinoid open subset of $D_{\mathbb{C}_\infty}(0,\rho)\backslash\{0\}$. We have $\mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho)\backslash\{0\}) = \bigcap_{0<\sigma\leqslant\rho} \mathcal{O}(A_{\mathbb{C}_\infty}[\sigma,\rho])$. Since $A_{\mathbb{C}_\infty}[\sigma,\rho]$ is reduced, the assumption implies that the restriction of f to this annulus is zero. Thus f itself is also zero. □

Lemma 8.8. *Let $k \geqslant 1$ be any integer. For any integers $r, s \geqslant 0$, the infinite sum*

$$E_k(X) = \sum_{(0,0)\neq(c,d)\in A^2} \frac{1}{(cX+d)^k}$$

converges in the affinoid algebra $\mathcal{O}(\Omega_{r,s})$. In particular, $E_k(X)$ defines a rigid analytic function on Ω.

Proof. Since $\Omega_{r,s}$ is reduced, [BGR, Theorem 6.2.4/1] implies that the Banach topology on $\mathcal{O}(\Omega_{r,s})$ is defined by the supremum norm. For any $z \in \Omega_{r,s}$, we have $|z|_i \geqslant q^{-r}$ and

$$\left|\frac{1}{cz+d}\right| \leqslant \begin{cases} q^{-\deg(d)} & (c = 0), \\ q^{r-\deg(c)} & (c \neq 0). \end{cases} \tag{8.4}$$

This implies

$$\left|\frac{1}{cX+d}\right|_{\sup} \to 0 \quad (\deg(c)+\deg(d) \to +\infty)$$

in $\mathcal{O}(\Omega_{r,s})$. Thus the infinite sum converges to $E_k(X) \in \mathcal{O}(\Omega_{r,s})$. Since the restriction map is continuous, the rigid analytic function $E_k(X)$ is independent of r, s and it defines an element of $\mathcal{O}(\Omega)$. □

Proposition 8.9. *Let $k \geqslant 1$ be any integer. Then $E_k \in M_{k,0}(GL_2(A))$. Moreover, we have*

$$\begin{cases} E_k = 0 & (k \not\equiv 0 \bmod q-1), \\ E_k \notin S_{k,0}(GL_2(A)) & (k \equiv 0 \bmod q-1). \end{cases}$$

In particular, we have $E_k \neq 0$ if $k \equiv 0 \bmod q-1$.

Proof. First we show (6.13). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$. For any $(C, D) \in A^2 \backslash \{(0,0)\}$, we have

$$(C\gamma(z) + D)^{-k} = (cz+d)^k((aC + cD)z + (bC + dD))^{-k}.$$

Note that the map

$$A^2\backslash\{(0,0)\} \to A^2\backslash\{(0,0)\}, \quad (C,D) \mapsto (aC + cD, bC + dD) = (C,D)\gamma$$

is a bijection and thus

$$\sum_{(C,D)\in A^2\backslash\{(0,0)\}} ((aC+cD)z+(bC+dD))^{-k} = \sum_{(C,D)\in A^2\backslash\{(0,0)\}} (Cz+D)^{-k} = E_k(z).$$

This yields (6.13). In particular, as in the proof of Lemma 6.57 we have $E_k = 0$ unless $k \equiv 0 \bmod q-1$.

Next we assume $k \equiv 0 \bmod q-1$ and show that E_k is regular at cusps. Note that $GL_2(A)$ has the unique cusp, which is represented by ∞. Consider the Fourier expansion at ∞ for $\nu = \mathrm{id}$. Then Proposition 8.4 yields

$$\begin{aligned} E_k(z) &= \sum_{0\neq d\in A} d^{-k} + \sum_{c\in A_+}\sum_{d\in A}(cz+d)^{-k} \\ &= \sum_{0\neq d\in A} d^{-k} + \sum_{c\in A_+} G_{A,k}(\bar{\pi}u_A(cz)). \end{aligned} \tag{8.5}$$

By Lemma 8.6, for any sufficiently small integer r, there exists $F \in \mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho_r))$ such that the series (8.5) agrees with $F(u_A(z))$ for any $z \in \Omega_r$. On the other hand, the Fourier expansion at ∞ of E_k yields a rigid analytic function $G \in \mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho_r)\backslash\{0\})$ such that the series (8.5) agrees with $G(u_A(z))$ for any $z \in \Omega_r$. By (6.10) and Lemma 8.7, we obtain $F = G$, which shows that E_k is regular at ∞ and

$$(F(u) - \sum_{0\neq d\in A} d^{-k})|_{u=0} = 0.$$

Suppose $k \equiv 0 \bmod q-1$. To show $E_k \notin S_{k,0}(GL_2(A))$, it is enough to show

$$\sum_{0\neq d\in A} d^{-k} \neq 0.$$

For this, the assumption on k yields

$$\sum_{d \in \mathbb{F}_q^\times} d^{-k} = -1.$$

Since $|d^{-k}| = q^{-k\deg(d)} < 1$ for any $d \in A \backslash \mathbb{F}_q$, we obtain $|\sum_{0 \neq d \in A} d^{-k}| = 1$. This concludes the proof of the proposition. □

Lemma 8.10. *Let $Q \in A \backslash \mathbb{F}_q$ be a monic irreducible polynomial and let $c \in A$ be an element which is coprime to Q. Then the map*

$$A \times \{\beta \in A \mid \deg(\beta) < \deg(Q)\} \to A, \quad (d, \beta) \mapsto dQ + \beta c$$

is a bijection.

Proof. If (d, β) and (d', β') satisfies $dQ + \beta c = d'Q + \beta' c$, then $(\beta - \beta')c = (d' - d)Q$ and the assumption $Q \nmid c$ yields $\beta = \beta'$ and $d' = d$.

For the surjectivity, we can find $a, b \in A$ satisfying $aQ + bc = 1$. For any $f \in A$, we have $afQ + bfc = f$. Write $bf = RQ + \beta$ with some $R \in A$ and $\beta \in A$ satisfying $\deg(\beta) < \deg(Q)$. Then $(af + cR)Q + \beta c = f$ and the lemma follows. □

Lemma 8.11. *Let $Q \in A \backslash \mathbb{F}_q$ be a monic irreducible polynomial. Then*

$$T_Q E_k = Q^{k-1} E_k.$$

Proof. By Example 7.18, we have

$$\begin{aligned}(T_Q E_k)(z) &= \frac{1}{Q} \sum_{\deg(\beta) < \deg(Q)} \sum_{(c,d) \neq (0,0)} \frac{1}{\left(c\left(\frac{z+\beta}{Q}\right) + d\right)^k} + Q^{k-1} \sum_{(c,d) \neq (0,0)} \frac{1}{(cQz + d)^k} \\ &= \sum_{\deg(\beta) < \deg(Q)} \sum_{(c,d) \neq (0,0)} \frac{Q^{k-1}}{(cz + (dQ + \beta c))^k} + \sum_{(c,d) \neq (0,0)} \frac{Q^{k-1}}{(cQz + d)^k}.\end{aligned}$$

For $Q \nmid c$ in the former sum, we have $c \neq 0$. By Lemma 8.10, we obtain

$$\sum_{\deg(\beta) < \deg(Q)} \sum_{Q \nmid c,\ d \in A} \frac{Q^{k-1}}{(cz + (dQ + \beta c))^k} = \sum_{Q \nmid c,\ d \in A} \frac{Q^{k-1}}{(cz + d)^k}.$$

On the other hand, for $Q \mid c$ in the former sum, write $c = QC$ and we have

$$\sum_{\deg(\beta)<\deg(Q)} \sum_{(c,d)\neq(0,0),\ Q|c} \frac{Q^{k-1}}{(cz+(dQ+\beta c))^k}$$
$$= \sum_{\deg(\beta)<\deg(Q)} \sum_{(C,d)\neq(0,0)} \frac{Q^{k-1}}{(QCz+(dQ+\beta QC))^k}$$
$$= \sum_{\deg(\beta)<\deg(Q)} \sum_{(C,d)\neq(0,0)} \frac{Q^{-1}}{(Cz+(d+\beta C))^k}.$$

Since the map

$$A^2\backslash\{(0,0)\} \to A^2\backslash\{(0,0)\}, \quad (C,d) \mapsto (C, d+\beta C)$$

is a bijection, the sum equals

$$\sum_{\deg(\beta)<\deg(Q)} \sum_{(C,d)\neq(0,0)} \frac{Q^{-1}}{(Cz+d)^k} = q^{\deg(Q)} \sum_{(C,d)\neq(0,0)} \frac{Q^{-1}}{(Cz+d)^k} = 0.$$

Hence we obtain

$$(T_Q E_k)(z) = \sum_{Q\nmid c,\ d\in A} \frac{Q^{k-1}}{(cz+d)^k} + \sum_{(c,d)\neq(0,0)} \frac{Q^{k-1}}{(cQz+d)^k} = Q^{k-1}E_k(z).$$

This concludes the proof. □

8.3. **Poincaré series.**

Lemma 8.12. *For any* $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$ *and* $z \in \Omega$, *put*

$$j(\xi, z) = cz + d.$$

Then we have

$$j(\xi\xi', z) = j(\xi, \xi'(z))j(\xi', z) \quad \textit{for any } \xi, \xi' \in GL_2(K).$$

Proof. Put $\xi' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then the lemma follows from

$$j(\xi, \xi'(z))j(\xi', z) = c(a'z+b') + d(c'z+d') = j(\xi\xi', z).$$

□

Let

$$H = \left\{ \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq GL_2(A).$$

Note that we have a bijection

$$H\backslash GL_2(A) \to \{(c,d) \in A^2 \mid cA + dA = A\}, \quad \gamma \mapsto (0,1)\gamma.$$

Lemma 8.13. *Let k, m be integers satisfying $k \geqslant 1$. For any $\gamma \in GL_2(A)$, the element*

$$\frac{\det(\gamma)^m}{j(\gamma, X)^k} u_A(\gamma(X))^m \in \mathcal{O}(\Omega)$$

depends only on the class of γ in $H \backslash GL_2(A)$.

Proof. Since Ω is reduced, we may check the independence pointwise. Take any $z \in \Omega$ and $\gamma \in GL_2(A)$. Put $h = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{F}_q^\times$ and $b \in A$. Then we have

$$\exp_A(h\gamma(z)) = \exp_A(a\gamma(z)+b) = a\exp_A(\gamma(z))+\exp_A(b) = a\exp_A(\gamma(z)).$$

On the other hand, Lemma 8.12 yields

$$j(h\gamma, z) = j(h, \gamma(z))j(\gamma, z) = j(\gamma, z).$$

Since $\det(h) = a$, the lemma follows. $\square$

Lemma 8.14. *For any integers k, m satisfying $k \geqslant 1$, the infinite sum*

$$\sum_{\gamma \in H \backslash GL_2(A)} \frac{\det(\gamma)^m}{j(\gamma, X)^k} u_A(\gamma(X))^m \tag{8.6}$$

converges and defines an element $P_{k,m}(X) \in \mathcal{O}(\Omega)$.

Proof. Let r, s be any positive integers. By Corollary 6.46, the supremum norm on the affinoid variety $\Omega_{r,s}$ satisfies

$$\left| \frac{\det(\gamma)^m}{j(\gamma, X)^k} u_A(\gamma(X))^m \right|_{\sup} \leqslant \left| \frac{1}{j(\gamma, X)} \right|_{\sup}^k \cdot |\pi|^{-1} \sigma_{A,q^{-r}}^{-1}.$$

Then (8.4) implies that the infinite sum converges in $\mathcal{O}(\Omega_{r,s})$. $\square$

Lemma 8.15. *For any integers k, m satisfying $k \geqslant 1$ and $m \not\equiv 0 \bmod q-1$, we have*

$$P_{k,m} \in S_{k,m}(GL_2(A)).$$

Proof. For any $\delta \in GL_2(A)$, Lemma 8.12 yields

$$\begin{aligned}
(P_{k,m}|_{k,m}\delta)(z) &= \det(\delta)^m j(\delta, z)^{-k} P_{k,m}(\delta(z)) \\
&= \det(\delta)^m j(\delta, z)^{-k} \sum_{\gamma \in H \backslash GL_2(A)} \frac{\det(\gamma)^m}{j(\gamma, \delta(z))^k} u_A(\gamma\delta(z)))^m \\
&= \sum_{\gamma \in H \backslash GL_2(A)} \frac{\det(\gamma\delta)^m}{j(\gamma\delta, z)^k} u_A(\gamma\delta(z)))^m = P_{k,m}(z)
\end{aligned}$$

and the condition (6.13) follows. By (8.4), we see that $|P_{k,m}|_{\sup}$ on $\Omega_{r,s}$ is bounded independently of s. This implies that it is bounded on Ω_r and thus it is regular at ∞.

To show that $P_{k,m}$ vanishes at the cusp ∞, write its Fourier expansion at ∞ as

$$P_{k,m}(X) = a_0 + a_1 u_A(X) + a_2 u_A(X)^2 + \cdots, \quad a_i \in \mathbb{C}_\infty.$$

For any $c \in \mathbb{F}_q^\times$, the action of the matrix $\begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ yields $P_{k,m}(cX) = c^{-m} P_{k,m}(X)$, which forces $a_0 = 0$. □

We can prove the following non-vanishing result of $P_{k,m}$ as in the proof of [GvdP, Proposition 10.15.2].

Proposition 8.16. *Let k, m be integers. Suppose $k \geqslant 1$, $k \equiv 2m \bmod q-1$ and $0 \leqslant m \leqslant k/(q+1)$. Then $P_{k,m} \neq 0$.*

Proof. It is enough to show $P_{k,m}(\sqrt{t}) \neq 0$. For this, we divide the sum (8.6) into three partial sums of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \backslash GL_2(A)$ satisfying the following conditions:

(A) $c = 0$ and $d \in \mathbb{F}_q^\times$.
(B) $c \in \mathbb{F}_q^\times$ and $d \in \mathbb{F}_q$.
(C) $\deg(c) + \deg(d) \geqslant 1$.

Let $S_\bullet$ be the corresponding partial sum for $\bullet \in \{A, B, C\}$.

Note that for any $c \in \mathbb{F}_q^\times$ we have $u_A(cX) = c^{-1} u_A(X)$. For the case (A), for any $d \in \mathbb{F}_q^\times$ we may take $\gamma = \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}$. Then the assumption $k \equiv 2m \bmod q-1$ implies

$$S_A = \sum_{d \in \mathbb{F}_q^\times} \frac{1}{d^k} u_A(d^{-2}\sqrt{t})^m = \sum_{d \in \mathbb{F}_q^\times} d^{2m-k} u_A(\sqrt{t})^m = -u_A(\sqrt{t})^m \neq 0.$$

For any $\alpha \neq 0 \in A$ we have $|\frac{\sqrt{t}}{\alpha}| = q^{\frac{1}{2} - \deg(\alpha)}$, and thus $|\frac{\sqrt{t}}{\alpha}| \geqslant 1$ if and only if $\alpha \in \mathbb{F}_q^\times$. This yields

$$|\exp_A(\sqrt{t})| = |\sqrt{t}| \prod_{\alpha \in \mathbb{F}_q^\times} \left| 1 - \frac{\sqrt{t}}{\alpha} \right| = |\sqrt{t}|^q$$

and we obtain

$$|S_A| = \frac{1}{|\sqrt{t}|^{qm} |\bar{\pi}|^m}. \tag{8.7}$$

For the case (B), for any $(c,d) \in \mathbb{F}_q^\times \times \mathbb{F}_q$ we may take $\gamma = \begin{pmatrix} 0 & -c^{-1} \\ c & d \end{pmatrix}$. By the assumption $k \equiv 2m \bmod q-1$, we have

$$\begin{aligned}
S_B &= \sum_{(c,d)\in\mathbb{F}_q^\times\times\mathbb{F}_q} \frac{1}{(c\sqrt{t}+d)^k} u_A\left(\frac{-c^{-1}}{c\sqrt{t}+d}\right)^m \\
&= \sum_{(c,d)\in\mathbb{F}_q^\times\times\mathbb{F}_q} (-1)^m c^{2m-k} \frac{1}{(\sqrt{t}+c^{-1}d)^k} u_A\left(\frac{1}{\sqrt{t}+c^{-1}d}\right)^m \\
&= \sum_{c\in\mathbb{F}_q^\times} (-1)^m c^{2m-k} \sum_{d\in\mathbb{F}_q} \frac{1}{(\sqrt{t}+d)^k} u_A\left(\frac{1}{\sqrt{t}+d}\right)^m \\
&= (-1)^{m+1} \sum_{d\in\mathbb{F}_q} \frac{1}{(\sqrt{t}+d)^k} u_A\left(\frac{1}{\sqrt{t}+d}\right)^m.
\end{aligned}$$

For any $d \in \mathbb{F}_q$ and $\alpha \neq 0 \in A$, we have $|\sqrt{t}+d| = q^{\frac{1}{2}}$ and

$$\left|\frac{1}{\alpha(\sqrt{t}+d)}\right| = q^{-\frac{1}{2}-\deg(\alpha)} < 1.$$

This yields

$$\exp_A\left(\frac{1}{\sqrt{t}+d}\right) = \frac{1}{\sqrt{t}+d}(1+\delta), \quad |\delta| < 1$$

and thus

$$u_A\left(\frac{1}{\sqrt{t}+d}\right) = \frac{1}{\bar{\pi}}(\sqrt{t}+d)(1+\delta'), \quad |\delta'| < 1.$$

Since $|d| < |\sqrt{t}|$, with some $\delta'' \in \mathbb{C}_\infty$ satisfying

$$|\delta''| < \frac{1}{|\sqrt{t}+d|^{k-m}|\bar{\pi}|^m} = \frac{1}{|\sqrt{t}|^{k-m}|\bar{\pi}|^m}, \tag{8.8}$$

we have

$$\begin{aligned}
S_B &= (-1)^{m+1} \sum_{d\in\mathbb{F}_q} \frac{1}{(\sqrt{t}+d)^{k-m}\bar{\pi}^m}(1+\delta')^m \\
&= (-1)^{m+1} \sum_{d\in\mathbb{F}_q} \frac{1}{\sqrt{t}^{k-m}\bar{\pi}^m} + \delta'' = \delta''.
\end{aligned}$$

Now the assumption $m \leqslant \frac{k}{q+1}$ yields $qm \leqslant k-m$ and

$$\left|\frac{1}{\sqrt{t}}\right|^{k-m} \leqslant \left|\frac{1}{\sqrt{t}}\right|^{qm},$$

which shows $|S_B| < |S_A|$.

Let us consider the case C. For any $(c,d)\in A^2$ satisfying $cA+dA=A$ and $\deg(c)+\deg(d)\geqslant 1$, we may take $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(A)$ such that $\deg(a)<\deg(c)$ and $\deg(b)<\deg(d)$. Indeed, given $a,b\in A$ satisfying $ad-bc=1$, write $a=sc+r$ with $r,s\in A$ with $\deg(r)<\deg(c)$. Then we have $rd-(b-sd)c=1$ and replacing a with r we may assume $\deg(a)<\deg(c)$. Note that the inequality $\deg(c)+\deg(d)\geqslant 1$ yields $cd\neq 0$. Since we have $ad-bc=1$, if $a=0$ then $bc=-1$ and we may take any $d\in A\backslash\mathbb{F}_q$ so that $\deg(b)<\deg(d)$. Similarly, if $b=0$ then $ad=1$, and in this case the inequality $\deg(b)<\deg(d)$ holds. If $ab\neq 0$, then we have $\deg(ad)=\deg(bc)$ and thus we obtain $\deg(b)<\deg(d)$.

Then we have $|a|\leqslant |c|q^{-1}$ and $|b|\leqslant |d|q^{-1}$. Note that the equality $|a\sqrt{t}|=|b|$ never holds since it would imply $a=b=0$. Thus we have $|a\sqrt{t}+b|=\max\{|a|q^{\frac{1}{2}},|b|\}$ and

$$\left|\frac{a\sqrt{t}+b}{c\sqrt{t}+d}\right|=\frac{\max\{|a|q^{\frac{1}{2}},|b|\}}{\max\{|c|q^{\frac{1}{2}},|d|\}}\leqslant\frac{\max\{|c|q^{\frac{-1}{2}},|d|q^{-1}\}}{\max\{|c|q^{\frac{1}{2}},|d|\}}=q^{-1}<1.$$

Hence we obtain

$$\left|\exp_A\left(\frac{a\sqrt{t}+b}{c\sqrt{t}+d}\right)\right|=\left|\frac{a\sqrt{t}+b}{c\sqrt{t}+d}\right|,\quad \left|u_A\left(\frac{a\sqrt{t}+b}{c\sqrt{t}+d}\right)\right|=\frac{|c\sqrt{t}+d|}{|\bar{\pi}||a\sqrt{t}+b|}$$

and thus

$$|S_C|\leqslant\frac{1}{|c\sqrt{t}+d|^{k-m}|a\sqrt{t}+b|^m|\bar{\pi}|^m}.$$

Now we have $|a\sqrt{t}+b|=\max\{|a|q^{\frac{1}{2}},|b|\}\geqslant 1$ and the assumption $\deg(c)+\deg(d)\geqslant 1$ yields $|c\sqrt{t}+d|=\max\{|c|q^{\frac{1}{2}},|d|\}>q^{\frac{1}{2}}=|\sqrt{t}|$. By the assumption $0\leqslant m\leqslant k/(q+1)$, we obtain

$$|S_C|<\frac{1}{|\sqrt{t}|^{k-m}|\bar{\pi}|^m}\leqslant\frac{1}{|\sqrt{t}|^{qm}|\bar{\pi}|^m}=|S_A|,$$

which yields

$$|P_{k,m}(\sqrt{t})|=|S_A+S_B+S_C|=|S_A|\neq 0.$$

This concludes the proof of the proposition. □

Definition 8.17. By Proposition 8.16, it follows that $h:=P_{q+1,1}$ is a nonzero element of $S_{q+1,1}(GL_2(A))$. We call it Gekeler's h-function.

8.4. **Petrov's family.** Let $E_{2k}(z)$ be the classical Eisenstein series of weight $2k$. It has the Lambert expansion

$$E_{2k}(z)=1+\frac{2}{\zeta(1-2k)}\sum_{n\geqslant 1}n^{2k-1}\frac{q^n}{1-q^n},\quad q=\exp(2\pi\sqrt{-1}z).$$

Put $q_n = q^n$ and $G(x) = \frac{x}{1-x}$. Then the expansion above is written as

$$E_{2k}(z) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n \geqslant 1} n^{2k-1} G(q_n).$$

Petrov [Pet] gave a family of Drinfeld cuspforms of level $GL_2(A)$ admitting an expansion in a similar spirit, which is called A-expansion. In this subsection we explain his construction.

Let v_p be the p-adic additive valuation normalized as $v_p(p) = 1$. For any integer d, put

$$A_{<d} = \{a \in A \mid \deg(a) < d\}.$$

Let $\bar{\pi}$ be the Carlitz period we fixed in Definition 6.16. For any positive integer n, let $G_{A,n}(X)$ be the n-th Goss polynomial for A and put

$$G_n(X) := \bar{\pi}^{-n} G_{A,n}(\bar{\pi} X).$$

By Proposition 8.4, we have

$$G_n(u_A(z)) = \bar{\pi}^{-n} G_{A,n}(\exp_A(z)^{-1}) = \bar{\pi}^{-n} \sum_{a \in A} \frac{1}{(z+a)^n}. \tag{8.9}$$

Proposition 8.18. *Let k, n be positive integers satisfying $k - 2n \in (q-1)\mathbb{Z}_{>0}$ and $n \leqslant p^{v_p(k-n)}$ so that $k > 2n$. Then the series*

$$f_{k,n}(z) := \sum_{a \in A_+} a^{k-n} G_n(u_A(az))$$

converges to define an element of $S_{k,m}(GL_2(A))$.

Proof. First note that the condition $n \leqslant p^{v_p(k-n)}$ implies that $(T-1)^n$ divides $(T^{k-n} - 1)$ in the polynomial ring $\mathbb{F}_p[T]$. Thus we may define

$$F(T) := \frac{T^{k-n} - 1}{(T-1)^n} = \sum_{i=0}^{k-2n} \xi_i T^i, \quad \xi_i \in \mathbb{F}_p.$$

Note that if $k = 2n$, then the inequality $n \leqslant p^{v_p(k-n)} \leqslant k - n = n$ implies that $n = k - n$ is a p-power integer and $F(T) = 1$.

Lemma 8.19. *For any integers $d \geqslant 2$ and $j \in [1, d-1]$, we have*

$$S_{d,j} := \sum_{a \in A_{<d}} a^j = \sum_{a \neq 0 \in A_{<d}} a^j = 0.$$

Proof. Note that $A_{<d}$ is an $\mathbb{F}_q$-vector space of dimension d. Let $a_1, \dots, a_d$ be its basis. Write

$$\begin{aligned} S_{d,j} &= \sum_{c_1 \in \mathbb{F}_q} \cdots \sum_{c_d \in \mathbb{F}_q} (c_1 a_1 + \cdots + c_d a_d)^j \\ &= \sum_{c_1 \in \mathbb{F}_q} \cdots \sum_{c_d \in \mathbb{F}_q} \sum_{i_1 + \cdots + i_d = j} \frac{j!}{i_1! \cdots i_d!} (c_1 a_1)^{i_1} \cdots (c_d a_d)^{i_d} \\ &= \sum_{i_1 + \cdots + i_d = j} \sum_{c_1 \in \mathbb{F}_q} \cdots \sum_{c_d \in \mathbb{F}_q} \frac{j!}{i_1! \cdots i_d!} (c_1 a_1)^{i_1} \cdots (c_d a_d)^{i_d}. \end{aligned}$$

Since $j < d$, for any $(i_1, \dots, i_d) \in \mathbb{Z}_{\geqslant 0}$ satisfying $i_1 + \cdots + i_d = j$ we have $i_m = 0$ with some $m \in [1, d]$. Thus the sum over c_m of the term of $(i_1, \dots, i_d)$ is zero, which yields the lemma. $\square$

Lemma 8.20. *Let $z \in \Omega$. If $d \geqslant k - 2n + 1$, then*

$$\sum_{(u,v) \neq (0,0) \in A_{<d}^2} \frac{(vz)^{k-n} - u^{k-n}}{(vz - u)^n} = 0.$$

Proof. Put $f(u, v, z) = \frac{(vz)^{k-n} - u^{k-n}}{(vz-u)^n}$. Since $k > 2n$ and $d \geqslant k - 2n + 1$, Lemma 8.19 yields

$$\begin{aligned} \sum_{v \neq 0 \in A_{<d}} f(0, v, z) &= z^{k-2n} \sum_{v \neq 0 \in A_{<d}} v^{k-2n} = z^{k-2n} S_{d,k-2n} = 0, \\ \sum_{u \neq 0 \in A_{<d}} f(u, 0, z) &= (-1)^{n+1} \sum_{u \neq 0 \in A_{<d}} u^{k-2n} = (-1)^{n+1} S_{d,k-2n} = 0. \end{aligned}$$

Note that if $u \neq 0$ and $v \neq 0$, then we have

$$f(u, v, z) = u^{k-2n} \frac{(\frac{vz}{u})^{k-n} - 1}{(\frac{vz}{u} - 1)^n} = u^{k-2n} F\left(\frac{vz}{u}\right).$$

This implies

$$\begin{aligned} S := \sum_{u \neq 0 \in A_{<d}} \sum_{v \neq 0 \in A_{<d}} f(u, v, z) &= \sum_{u \neq 0 \in A_{<d}} \sum_{v \neq 0 \in A_{<d}} u^{k-2n} F\left(\frac{vz}{u}\right) \\ &= \sum_{u \neq 0 \in A_{<d}} \sum_{v \neq 0 \in A_{<d}} \sum_{i=0}^{k-2n} \xi_i u^{k-2n-i} (vz)^i \\ &= \sum_{u \neq 0 \in A_{<d}} \sum_{i=0}^{k-2n} \xi_i u^{k-2n-i} z^i S_{d,i}. \end{aligned}$$

Since $d \geqslant k-2n+1$, Lemma 8.19 yields $S_{d,i} = 0$ for any $i \in [1, k-2n]$ and

$$S = \sum_{u \neq 0 \in A_{<d}} \xi_0 u^{k-2n} S_{d,0} = \xi_0 S_{d,0} S_{d,k-2n} = 0.$$

This concludes the proof. □

Lemma 8.21. *Suppose $d \geqslant k-2n+1$. Then, for any $(a,b) \in t^d A^2 \backslash \{(0,0)\}$, we have*

$$\sum_{(u,v)\in A^2_{<d}} \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n} = \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n}.$$

Proof. Suppose $(u,v) \neq (0,0)$. By the equality

$$X^{k-n} - Y^{k-n} = (X-Y)^n Y^{k-2n} F\left(\frac{X}{Y}\right),$$

the difference

$$\frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n} - \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n}$$

equals

$$\frac{(bu-av)^{k-n} - ((a+u)(az+b))^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n} \tag{8.10}$$
$$= \frac{((bu-av)-(a+u)(az+b))^n}{(az+b)^{k-n}((a+u)z+b+v)^n} \cdot \sum_{i=0}^{k-2n} \xi_i (bu-av)^i ((a+u)(az+b))^{k-2n-i}.$$

Since $(bu-av)-(a+u)(az+b) = -a((a+u)z+b+v)$, (8.10) is equal to

$$\frac{(-a)^n}{(az+b)^{k-n}} \cdot \sum_{i=0}^{k-2n} \xi_i (bu-av)^i ((a+u)(az+b))^{k-2n-i}.$$

For $i \in [1, k-2n]$, Lemma 8.19 yields

$$\begin{aligned}
&\sum_{(u,v)\neq(0,0)\in A_{<d}^2} (bu-av)^i(a+u)^{k-2n-i}\\
&= \sum_{u\neq 0\in A_{<d}} (bu)^i(a+u)^{k-2n-i} + \sum_{v\neq 0\in A_{<d}} (-av)^i a^{k-2n-i}\\
&\quad + \sum_{u\neq 0\in A_{<d}} \sum_{v\neq 0\in A_{<d}} \sum_{j=0}^{i} \binom{i}{j} (bu)^{i-j}(-av)^j(a+u)^{k-2n-i}\\
&= \sum_{u\neq 0\in A_{<d}} (bu)^i(a+u)^{k-2n-i} + (-1)^i a^{k-2n} S_{d,i}\\
&\quad + \sum_{u\neq 0\in A_{<d}} \sum_{j=0}^{i} \binom{i}{j} (bu)^{i-j}(-a)^j(a+u)^{k-2n-i} S_{d,j}\\
&= \sum_{u\neq 0\in A_{<d}} (bu)^i(a+u)^{k-2n-i} + \sum_{u\neq 0\in A_{<d}} (bu)^i(a+u)^{k-2n-i} S_{d,0}\\
&= (1+S_{d,0}) \sum_{u\neq 0\in A_{<d}} \sum_{j=0}^{k-2n-i} \binom{k-2n-i}{j} b^i a^{k-2n-i-j} u^{i+j}\\
&= (1+S_{d,0}) \sum_{j=0}^{k-2n-i} \binom{k-2n-i}{j} b^i a^{k-2n-i-j} S_{d,i+j} = 0.
\end{aligned}$$

For $i = 0$, we have

$$\begin{aligned}
&\sum_{(u,v)\neq(0,0)\in A_{<d}^2} \frac{(-a)^n}{(az+b)^{k-n}} \xi_0((a+u)(az+b))^{k-2n}\\
&= \frac{(-a)^n}{(az+b)^n}\xi_0 \sum_{(u,v)\neq(0,0)\in A_{<d}^2} (a+u)^{k-2n}\\
&= \frac{(-a)^n}{(az+b)^n}\xi_0 \left(\sum_{u\neq 0\in A_{<d}} \sum_{v\in A_d} (a+u)^{k-2n} + \sum_{v\neq 0\in A_{<d}} a^{k-2n} \right)\\
&= \frac{(-a)^n}{(az+b)^n}\xi_0(-a^{k-2n}).
\end{aligned}$$

Since $T^{k-n} - 1 = (T-1)^n F(T)$, we have $\xi_0 = F(0) = (-1)^{n+1}$ and

$$\frac{(-a)^n}{(az+b)^n}\xi_0(-a^{k-2n}) = \frac{a^{k-n}}{(az+b)^n},$$

which agrees with the term on the left-hand side of the lemma for $(u,v) = (0,0)$. This concludes the proof. $\square$

Now choose any integer $d \geqslant k-2n+1$ and define

(8.11)

$$\phi_{k,n}(z) := \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \frac{u^{k-n}}{(uz+v)^n} + \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \sum_{(a,b)\neq(0,0)\in t^d A^2} \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n}.$$

Lemma 8.22. *For any $z \in \Omega$, the series (8.11) converges and defines an element $\phi_{k,n} \in \mathcal{O}(\Omega)$.*

Proof. Take any non-negative integers r, s. For any $(u,v) \neq (0,0) \in A^2_{<d}$ and any integer $m > d$, put

$$f_{u,v,a,b}(z) := \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n},$$
$$\phi_m(z) := \sum_{(a,b)\neq(0,0)\in t^d A^2_{<m-d}} f_{u,v,a,b}(z).$$

Then Lemma 5.26 implies $\phi_m \in \mathcal{O}(\Omega)$. By Proposition 5.24, it is enough to show that the sequence $\{\phi_m\}_{m>d}$ is Cauchy with respect to the supremum norm $|-|_{\mathrm{sup}}$ of the reduced affinoid variety $\Omega_{r,s}$.

Note that we have

$$\phi_{m+1}(z) - \phi_m(z) = \sum_{(a,b)\in t^d A^2_{<m+1-d} \backslash t^d A^2_{<m-d}} f_{u,v,a,b}(z)$$

and $(a,b) \in t^d A^2_{<m+1-d}$ lies in $t^d A^2_{<m+1-d} \backslash t^d A^2_{<m-d}$ exactly when one of the following condition holds:

(1) $\deg(a) = m \geqslant \deg(b)$.
(2) $\deg(a) < \deg(b) = m$.

For the case (1), we have $a \neq 0$ and $a+u \neq 0$. Since $z \in \Omega_{r,s}$, we have

$$|az+b| = |a|\left|z+\frac{b}{a}\right| \geqslant q^{m-r}, \quad |(a+u)z+b+v| = |a+u|\left|z+\frac{b+v}{a+u}\right| \geqslant q^{m-r}.$$

Thus

$$|f_{u,v,a,b}(z)|_{\mathrm{sup}} \leqslant \frac{q^{(d-1+m)(k-n)}}{q^{(m-r)k}} = q^{-mn+(d-1)(k-n)+rk}. \tag{8.12}$$

For the case (2), we have $|b+v| = |b| = q^m$. If $|az+b| \geqslant |b|$, then $|az+b| \geqslant q^m$. If $|az+b| < |b|$, then $a \neq 0$ and $|az| = |b|$. Since $z \in \Omega_{r,s}$, we have $q^{-r} \leqslant |z| \leqslant q^s$ and thus $|a| = |b||z|^{-1} \geqslant q^{m-s}$. This yields

$$|az+b| = |a| \left| z + \frac{b}{a} \right| \geqslant q^{m-s-r}.$$

Similar estimates hold for $|(a+u)z+b+v|$, which implies

$$\min\{|az+b|, |(a+u)z+b+z|\} \geqslant \min\{q^m, q^{m-s-r}\}.$$

Hence we obtain
(8.13)

$$|f_{u,v,a,b}(z)|_{\sup} \leqslant \frac{q^{(d-1+m)(k-n)}}{\min\{q^m, q^{m-s-r}\}^k} = q^{-mn+(d-1)(k-n)-k\min\{0,-r-s\}}.$$

By (8.12) and (8.13), there exists a constant C which is independent of m satisfying

$$|\phi_{m+1}(z) - \phi_m(z)|_{\sup} \leqslant q^{-mn+C}.$$

Since $n > 0$, we have

$$\lim_{m\to\infty} |\phi_{m+1}(z) - \phi_m(z)|_{\sup} = 0$$

and the lemma follows. □

Lemma 8.23.

$$\phi_{k,n}\left(\frac{-1}{z}\right) = z^k \phi_{k,n}(z).$$

Proof. We have

$$\phi_{k,n}\left(\frac{-1}{z}\right) = \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \frac{u^{k-n}z^n}{(vz-u)^n} + \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \sum_{(a,b)\neq(0,0)\in t^dA^2} \frac{z^k(bu-av)^{k-n}}{(bz-a)^{k-n}((b+v)z-(a+u))^n}.$$

By Lemma 8.20, this equals

$$\sum_{(u,v)\neq(0,0)\in A^2_{<d}} \frac{v^{k-n}z^k}{(vz-u)^n} + \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \sum_{(a,b)\neq(0,0)\in t^dA^2} \frac{z^k(bu-av)^{k-n}}{(bz-a)^{k-n}((b+v)z-(a+u))^n}.$$

Replacing u by $-u$ and a by $-a$, we see that this equals

$$\sum_{(u,v)\neq(0,0)\in A^2_{<d}} \frac{v^{k-n}z^k}{(vz+u)^n} + \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \sum_{(a,b)\neq(0,0)\in t^dA^2} \frac{z^k(av-bu)^{k-n}}{(bz+a)^{k-n}((b+v)z+(a+u))^n}.$$

Since $(u,v)\mapsto(v,u)$ and $(a,b)\mapsto(b,a)$ give bijections on the index sets of the sums, this agrees with $z^k\phi_{k,n}(z)$. □

Lemma 8.24.

$$\phi_{k,n}(z) = \bar{\pi}^k \sum_{a\in A} a^{k-n} G_n(u_A(az)).$$

In particular, the function $\phi_{k,n}(z)$ is independent of the choice of d.

Proof. By dividing the double summation defining $\phi_{k,n}$ into the sum for $a=0$ and $a\neq 0$, we can write

$$\begin{aligned}\phi_{k,n}(z) = &\sum_{(u,v)\neq(0,0)\in A^2_{<d}} \frac{u^{k-n}}{(uz+v)^n} \\ &+ \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \sum_{b\neq 0\in t^dA} \frac{(bu)^{k-n}}{b^{k-n}(uz+b+v)^n} \\ &+ \sum_{(u,v)\neq(0,0)\in A^2_{<d}} \sum_{a\neq 0\in t^dA} \sum_{b\in t^dA} \frac{(bu-av)^{k-n}}{(az+b)^{k-n}((a+u)z+b+v)^n}.\end{aligned}$$

For the first and second partial sums, since we have

$$A_{<d} \sqcup (t^dA\backslash\{0\} + A_{<d}) = A,$$

by (8.9) the sum of these partial sums equals

$$\sum_{u\neq 0\in A_{<d}} \sum_{v\in A} \frac{u^{k-n}}{(uz+v)^n} = \bar{\pi}^n \sum_{a\neq 0\in A_{<d}} a^{k-n} G_n(u_A(az)). \tag{8.14}$$

By Lemma 8.21, the third partial sum equals

$$\sum_{(u,v)\in A^2_{<d}} \sum_{a\neq 0\in t^dA} \sum_{b\in t^dA} \frac{(a+u)^{k-n}}{((a+u)z+b+v)^n}.$$

Since we have

$$A_{<d} + t^dA\backslash\{0\} = A\backslash A_{<d}, \quad A_{<d} + t^dA = A,$$

(8.9) implies that the sum equals

$$\sum_{a\in A\backslash A_{<d}}\sum_{b\in A}\frac{a^{k-n}}{(az+b)^n}=\bar{\pi}^n\sum_{a\in A\backslash A_{<d}}a^{k-n}G_n(u_A(az)). \tag{8.15}$$

Now the lemma follows from (8.14) and (8.15). □

Lemma 8.25. *For any $\gamma\in GL_2(A)$, we have $\phi_{k,n}|_{k,n}\gamma=\phi_{k,n}$.*

Proof. Note that by the theory of Smith normal forms, the group $GL_2(A)$ is generated by the elements

$$\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad\begin{pmatrix}1&b\\0&1\end{pmatrix}\ (b\in A),\quad\begin{pmatrix}c&0\\0&1\end{pmatrix}\ (c\in\mathbb{F}_q^\times).$$

For the first one, the equality of the lemma follows from Lemma 8.23. Since

$$\exp_A(a(z+b))=\exp_A(az)+\exp_A(ab)=\exp_A(az)\quad\text{for any } a,b\in A,$$

Lemma 8.24 yields the case of elements of the second kind. For the third one, since $(u,v)\mapsto(cu,v)$ and $(a,b)\mapsto(ca,b)$ give permutations on $A^2_{<d}\backslash\{(0,0)\}$ and $(t^dA)^2\backslash\{(0,0)\}$, by the definition of $\phi_{k,n}$ and the assumption $k-2n\in(q-1)\mathbb{Z}_{>0}$ we have

$$\phi_{k,n}(cz)=c^{n-k}\phi_{k,n}(z)=c^{-n}\phi_{k,n}(z).$$

This concludes the proof. □

By Lemma 8.6 and Lemma 8.24, for any sufficiently small integer r, there exists $F\in\mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho_r))$ satisfying $F(0)=0$ and $\phi_{k,n}(z)=F(u_A(z))$ for any $z\in\Omega_r$. On the other hand, the Fourier expansion at ∞ yields a rigid analytic function $G\in\mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho_r)\backslash\{0\})$ such that $\phi_{k,n}(z)=G(u_A(z))$ for any $z\in\Omega_r$. By (6.10) and Lemma 8.7, we obtain $F=G$, which shows that $\phi_{k,n}$ vanishes at ∞ and $\phi_{k,n}\in S_{k,n}(GL_2(A))$. Then Proposition 8.18 follows by putting

$$f_{k,n}:=\bar{\pi}^{-k}\phi_{k,n}.$$

□

Lemma 8.26.

$$f_{k,n}\neq 0.$$

Proof. As we have seen in the last paragraph of the proof of Proposition 8.18, Lemma 8.6 and Lemma 8.7 imply that the Fourier expansion $f_{k,n}(u)$ of $f_{k,n}$ at ∞ is given by the limit of the series

$$\sum_{a\in A_+}a^{k-n}G_n(f_a(u))$$

in $\mathcal{O}(D_{\mathbb{C}_\infty}(0,\rho))$ with some ρ. By Proposition 8.4, we can write as

$$G_n(X) = g_m X^m + \cdots + g_{n-1}X^{n-1} + X^n, \quad g_i \in \mathbb{C}_\infty,\ g_m \neq 0$$

with some integer $m \in [1,n]$. By Lemma 6.40, for any $a \in A_+$ with $\deg(a) > 0$, we have

$$f_a(u) \in u^q \mathbb{C}_\infty[[u]], \quad G_n(f_a(u)) \in u^{mq}\mathbb{C}_\infty[[u]].$$

Since $G_n(f_1(u)) = G_n(u) \in g_m u^m + u^{m+1}\mathbb{C}_\infty[u]$, this yields

$$f_{k,n}(u) \equiv g_m u^m \bmod u^{m+1}\mathbb{C}_\infty[[u]].$$

Hence $f_{k,n}$ has a nontrivial m-th coefficient in its Fourier expansion at ∞ and the lemma follows. □

Lemma 8.27. *Let k,n be positive integers satisfying $k-2n \in (q-1)\mathbb{Z}_{>0}$ and $n \leqslant p^{v_p(k-n)}$. Let $f_{k,n} \in S_{k,n}(GL_2(A))$ be the Drinfeld cuspform of Proposition 8.18. Let $Q \in A\backslash\mathbb{F}_q$ be a monic irreducible polynomial. Then*

$$T_Q f_{k,n} = Q^{n-1} f_{k,n}.$$

Proof. By Lemma 7.17 and Example 7.18, we have

$$\begin{aligned}(T_Q f_{k,n})(z) = {} & Q^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{a\in A_+} a^{k-n} G_n\left(u_A\left(\frac{a(z+\beta)}{Q}\right)\right) \\ & + Q^{k-1}\sum_{a \in A_+} a^{k-n} G_n(u_A(Qaz)).\end{aligned}$$

For the former sum, (8.9) yields

$$\begin{aligned}& Q^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{a\in A_+} a^{k-n} G_n\left(u_A\left(\frac{a(z+\beta)}{Q}\right)\right) \\ & = (\bar{\pi}^n Q)^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{a\in A_+}\sum_{b\in A} \frac{a^{k-n}Q^n}{(a(z+\beta)+Qb)^n}.\end{aligned}$$

When $Q \nmid a$, by Lemma 8.10 the map

$$A \times \{\beta \in A \mid \deg(\beta) < \deg(Q)\} \to A, \quad (b,\beta) \mapsto Qb + a\beta$$

is a bijection. Thus by (8.9) we obtain

$$
\begin{aligned}
&(\bar{\pi}^n Q)^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{a\in A_+,\ Q\nmid a} \sum_{b\in A} \frac{a^{k-n}Q^n}{(a(z+\beta)+Qb)^n}\\
&= (\bar{\pi}^n Q)^{-1} \sum_{a\in A_+,\ Q\nmid a} \sum_{b\in A} \frac{a^{k-n}Q^n}{(az+b)^n}\\
&= Q^{n-1} \sum_{a\in A_+,\ Q\nmid a} a^{k-n} G_n(u_A(az)).
\end{aligned}
$$

When $Q \mid a$, write $a = QC$ and we have

$$
\begin{aligned}
&= (\bar{\pi}^n Q)^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{a\in A_+,\ Q\mid a} \sum_{b\in A} \frac{a^{k-n}Q^n}{(a(z+\beta)+Qb)^n}\\
&= (\bar{\pi}^n Q)^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{C\in A_+} \sum_{b\in A} \frac{(QC)^{k-n}Q^n}{((QC)(z+\beta)+Qb)^n}\\
&= (\bar{\pi}^n Q)^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{C\in A_+} \sum_{b\in A} \frac{(QC)^{k-n}}{(Cz+C\beta+b)^n}.
\end{aligned}
$$

For any $\beta \in A$ with $\deg(\beta) < \deg(Q)$ and $C \in A_+$, the map

$$A \to A, \quad b \to C\beta + b$$

is a bijection. Thus the sum equals

$$(\bar{\pi}^n Q)^{-1} \sum_{\deg(\beta)<\deg(Q)} \sum_{C\in A_+} \sum_{b\in A} \frac{(QC)^{k-n}}{(Cz+b)^n} = 0.$$

Hence we obtain

$$
\begin{aligned}
(T_Q f_{k,n})(z) &= Q^{n-1} \sum_{a\in A_+,\ Q\nmid a} a^{k-n} G_n(u_A(az)) + Q^{k-1} \sum_{a\in A_+} a^{k-n} G_n(u_A(Qaz))\\
&= Q^{n-1} \sum_{a\in A_+,\ Q\nmid a} a^{k-n} G_n(u_A(az)) + Q^{n-1} \sum_{a\in A_+,\ Q\mid a} a^{k-n} G_n(u_A(az))\\
&= Q^{n-1} f_{k,n}(z).
\end{aligned}
$$

This concludes the proof. □

Remark 8.28. Fix a positive integer n. For any positive integer m, put $k = n + q^n(n + (q-1)m)$. Then

$$\frac{k-2n}{q-1} = \left(n\frac{q^n-1}{q-1} + q^n m \right) \in \mathbb{Z}_{>0}, \quad v_p(k-n) \geqslant n.$$

Thus Lemma 8.27 shows that $\{f_{n+q^n(n+(q-1)m),n}\}_{m\in\mathbb{Z}_{>0}}$ gives an infinite family of nonzero Drinfeld cuspforms of level $SL_2(A)$ and different weights such that each member of the family has the same Hecke eigenvalue for any monic irreducible polynomial $Q \in A$. In particular, a Drinfeld eigenform is not determined by its Hecke eigenvalues even up to a scalar multiple.

9. Harmonic cocycles

9.1. **Definition of harmonic cocycles.**

Definition 9.1. Let M be an additive group and let $\mathcal{T}$ be the Bruhat–Tits tree. A map $c : \mathcal{T}_1^o \to M$ is called a harmonic cocycle if the following conditions hold.

(1) For any $v \in \mathcal{T}_0$, we have
$$\sum_{e\in\mathcal{T}_1^o,\ t(e)=v} c(e) = 0.$$

(2) For any $e \in \mathcal{T}_1^o$, we have $c(-e) = -c(e)$.

The condition (1) is referred to as the harmonicity of c.

Definition 9.2. Let $V(\mathbb{C}_\infty) = \mathbb{C}_\infty^2$ be the set of row vectors with entries in $\mathbb{C}_\infty$. Let Γ be an arithmetic subgroup of $GL_2(K)$ and put
$$\begin{aligned} H_{k-2}(\mathbb{C}_\infty) &= \mathrm{Sym}^{k-2}(\mathrm{Hom}_{\mathbb{C}_\infty}(V(\mathbb{C}_\infty), \mathbb{C}_\infty)),\\ V_k(\mathbb{C}_\infty) &= \mathrm{Hom}_{\mathbb{C}_\infty}(H_{k-2}(\mathbb{C}_\infty), \mathbb{C}_\infty). \end{aligned}$$
They are endowed with natural left actions of Γ induced by its left action $\circ$ on $V(\mathbb{C}_\infty)$, which are also denoted by $\circ$. For any $P \in H_1(\mathbb{C}_\infty) = \mathrm{Hom}_{\mathbb{C}_\infty}(V(\mathbb{C}_\infty), \mathbb{C}_\infty)$ and $v \in V(\mathbb{C}_\infty)$, this means
$$(\gamma \circ P)(v) = P(\gamma^{-1} \circ v).$$

The action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$ on $H_{k-2}(\mathbb{C}_\infty)$ and $V_k(\mathbb{C}_\infty)$ is described as follows. Let $f_1 = (1, 0)$ and $f_2 = (0, 1)$ be the standard basis of $V(\mathbb{C}_\infty)$. Let $X = f_1^\vee$ and $Y = f_2^\vee$ be the dual basis of $H_1(\mathbb{C}_\infty)$. Then we have
$$\gamma\circ(f_1, f_2) = (f_1, f_2)^t\gamma^{-1}, \quad \gamma\circ(X, Y) = (X, Y)\gamma = (aX + cY, bX + dY).$$
We identify $H_{k-2}(\mathbb{C}_\infty)$ with the $\mathbb{C}_\infty$-subspace of the polynomial ring $\mathbb{C}_\infty[X, Y]$ consisting of polynomials of total degree $k - 2$. Then, for any $\omega \in V_k(\mathbb{C}_\infty)$, the action of γ is given by

(9.1)
$$\begin{aligned} (\gamma \circ \omega)(X^iY^{k-2-i}) &= \omega(\gamma^{-1} \circ X^iY^{k-2-i})\\ &= \det(\gamma)^{2-k}\omega((dX - cY)^i(-bX + aY)^{k-2-i}). \end{aligned}$$

Definition 9.3. Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$ be an integer. A harmonic cocycle $c : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ which is Γ-equivariant is called a harmonic cocycle of level Γ and weight k. The condition of being Γ-equivariant means

$$(\gamma \circ c)(e) = c(\gamma \circ e) \quad \text{for any } \gamma \in \Gamma,\ e \in \mathcal{T}_1^o.$$

The $\mathbb{C}_\infty$-vector space of harmonic cocycles of level Γ and weight k is denoted by $C_k^{\mathrm{har}}(\Gamma, \mathbb{C}_\infty)$ or $C_k^{\mathrm{har}}(\Gamma)$.

Definition 9.4. For any $c \in C_k^{\mathrm{har}}(\Gamma)$ and $\gamma \in GL_2(K)$, let

$$^{\gamma}c : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty), \quad ^{\gamma}c(e) = \gamma \circ c(\gamma^{-1} \circ e).$$

Then $^{\gamma}c \in C_k^{\mathrm{har}}(\gamma\Gamma\gamma^{-1})$. We have $^{\gamma}c = c$ for any $\gamma \in \Gamma$.

Proposition 9.5. *Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$ be an integer. Then any element $c \in C_k^{\mathrm{har}}(\Gamma)$ is cuspidal. Namely, there exists a finite subset S of $\Gamma\backslash\mathcal{T}_1^o$ such that $c(e) = 0$ if the Γ-equivalence class of e does not lie in S.*

Proof. Take any $\nu \in GL_2(K)$ satisfying $\nu\Gamma\nu^{-1} \subseteq GL_2(A)$. By replacing Γ by $\nu\Gamma\nu^{-1}$ and c by $^{\nu}c$, we may assume that Γ is a congruence subgroup of $GL_2(A)$. Moreover, by Lemma 3.12 and Lemma 3.15 it is enough to show that for any $g \in GL_2(A)$, we have $c(g \circ e_n) = 0$ for any sufficiently large integer n, where e_n is the standard edge of Definition 2.4. Again replacing c by $^{g^{-1}}c$ and Γ by $g^{-1}\Gamma g$, we may assume $g = \mathrm{id}$.

For this, by Lemma 3.4 there exists a nonzero element $\mathfrak{n} \in A$ of degree $d > 0$ such that $\Gamma(\mathfrak{n})$ is a subgroup of finite index of Γ. Since $C_k^{\mathrm{har}}(\Gamma) \subseteq C_k^{\mathrm{har}}(\Gamma(\mathfrak{n}))$, we may assume $\Gamma = \Gamma(\mathfrak{n})$.

Put

$$\begin{aligned} U &= \mathrm{Stab}_{\Gamma(\mathfrak{n})}(\infty) = \begin{pmatrix} 1 & \mathfrak{n}A \\ 0 & 1 \end{pmatrix}, \\ U_i &= \mathrm{Stab}_U(e_i) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathfrak{n}A,\ \deg(b) \leqslant i \right\} \quad (i \geqslant 1), \end{aligned}$$

where that last equality follows from (3.3). Then we have

$$U_i \subsetneq U_{i+1}\ (i \geqslant d), \quad U = \bigcup_{i \geqslant d} U_i.$$

Moreover, for any $i \geqslant d$ the quotient group U_{i+1}/U_i is isomorphic to the additive group $\mathbb{F}_q$.

Write $M = V_k(\mathbb{C}_\infty)$. For any $i \geqslant d$, let $M_i = M^{U_i}$ be the fixed part of M by the action $\circ$ of U_i. Since $M_{i+1} \subseteq M_i$ for any $i \geqslant d$ and M is finite-dimensional, there exists an integer $i_0 \geqslant d$ such that for any $i \geqslant i_0$ we have $M_i = M_{i+1}$.

Take any integer $j \geqslant i_0 + 1$. Since U_j fixes e_j and v_j, we have an injection

$$U_j/U_{j-1} \to \{e \in \mathcal{T}_1^o \mid t(e) = v_j\}\backslash\{-e_j\}, \quad \gamma \mapsto \gamma \circ e_{j-1}.$$

By comparing the cardinality we see that it is a bijection. Since c is Γ-equivariant, for any $\gamma \in U_{j-1}$ we have

$$\gamma \circ c(e_{j-1}) = c(\gamma \circ e_{j-1}) = c(e_{j-1})$$

and thus $c(e_{j-1}) \in M_{j-1} = M_j$. Then the harmonicity of c yields

$$\begin{aligned} c(e_j) &= \sum_{\gamma \in U_j/U_{j-1}} c(\gamma \circ e_{j-1}) = \sum_{\gamma \in U_j/U_{j-1}} \gamma \circ c(e_{j-1}) \\ &= \sum_{\gamma \in U_j/U_{j-1}} c(e_{j-1}) = qc(e_{j-1}) = 0. \end{aligned}$$

This concludes the proof. □

9.2. **Integration of polynomials via a harmonic cocycle.** Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$ be an integer. Let $c \in C_k^{\mathrm{har}}(\Gamma)$ be any harmonic cocycle of weight k and level Γ. We denote by P_k the $\mathbb{C}_\infty$-subspace of the polynomial ring $\mathbb{C}_\infty[x]$ consisting of polynomials of degree $\leqslant k-2$.

For any $e \in \mathcal{T}_1^o$ and any integer $0 \leqslant i \leqslant k-2$, define

$$\int_{U(e)} x^i d\mu_c(x) := (-1)^i c(e)(X^{k-2-i}Y^i), \tag{9.2}$$

where $U(e)$ is the distinguished closed disc in $\mathbb{P}^1(K_\infty)$ associated with the edge e as in Definition 4.17. By linearity, we obtain a $\mathbb{C}_\infty$-linear map

$$P_k \to \mathbb{C}_\infty, \quad f(x) \mapsto \int_{U(e)} f(x)d\mu_c(x).$$

For any $P(X,Y) \in H_{k-2}(\mathbb{C}_\infty)$, the equality (9.2) yields

$$c(e)(P(X,Y)) = \int_{U(e)} P(1,-x)d\mu_c(x).$$

Lemma 9.6. *Suppose that $e, e_1', \dots, e_r' \in \mathcal{T}_1^o$ satisfy*

$$U(e) = \coprod_{i=1}^r U(e_i').$$

Then we have

$$c(e) = \sum_{i=1}^r c(e_i'). \tag{9.3}$$

In particular, for any $f(x) \in P_k$ we have

$$\int_{U(e)} f(x) d\mu_c(x) = \sum_{i=1}^{r} \int_{U(e'_i)} f(x) d\mu_c(x).$$

Proof. By Lemma 4.26, for any half-line $H \in H(e)$ there exist a unique $i \in \{1, \ldots, r\}$ such that H passes though e'_i. Then the harmonicity of c yields (9.3). The second assertion of the lemma follows from (9.2). □

Let U be a compact open subset of $\mathbb{P}^1(K_\infty)$. By Lemma 4.13 and Lemma 4.2, we can write

$$U = \coprod_{i=1}^{r} U(e_i)$$

with some $e_i \in \mathcal{T}_1^o$. For any $f(x) \in P_k$, we put

$$\int_U f(x) d\mu_c(x) := \sum_{i=1}^{r} \int_{U(e_i)} f(x) d\mu_c(x).$$

It is independent of the choice of a decomposition of U into the disjoint union of distinguished closed discs in $\mathbb{P}^1(K_\infty)$. Indeed, by Lemma 4.16 we are reduced to the case of $U = U(e)$ with some $e \in \mathcal{T}_1^o$, which follows from Lemma 9.6.

Lemma 9.7. *(1) For any $f(x) \in P_k$, $e \in \mathcal{T}_1^o$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$, we have*

$$\int_{U(\gamma \circ e)} f(x) \mu_{\gamma c}(x) = \int_{U(e)} \det(\gamma)^{2-k} f(\gamma(x))(cx + d)^{k-2} d\mu_c(x).$$

(2) For any $f(x) \in P_k$, we have

$$\int_{\mathbb{P}^1(K_\infty)} f(x) d\mu_c(x) = 0.$$

Proof. Note that $f(\gamma(x))(cx+d)^{k-2} \in P_k$. By linearity, we may assume $f(x) = x^i$ with some $0 \leqslant i \leqslant k-2$. Then we have

$$\begin{aligned}\int_{U(\gamma\circ e)} f(x)\mu_{\gamma c}(x) &= (-1)^i {}^{\gamma}c(\gamma\circ e)(X^{k-2-i}Y^i)\\ &= (-1)^i \gamma\circ c(e)(X^{k-2-i}Y^i)\\ &= (-1)^i \det(\gamma)^{2-k} c(e)((dX-cY)^{k-2-i}(-bX+aY)^i)\\ &= \int_{U(e)} (-1)^i \det(\gamma)^{2-k}(d+cx)^{k-2-i}(-b-ax)^i d\mu_c(x)\\ &= \int_{U(e)} \det(\gamma)^{2-k}\left(\frac{ax+b}{cx+d}\right)^i (cx+d)^{k-2} d\mu_c(x).\end{aligned}$$

This proves (1).

For (2), since $\mathbb{P}^1(K_\infty)$ is compact we may compute the integral using the covering $\mathbb{P}^1(K_\infty) = U(e) \sqcup U(-e)$ for any $e \in \mathcal{T}_1^o$. Then Definition 9.1 (2) yields (2). $\square$

Lemma 9.8. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(K)$ *and let* $e \in \mathcal{T}_1^o$ *satisfying* $\infty \notin U(e)$ *and* $\infty \notin U(\gamma\circ e)$. *Then* $|cx+d|$ *is constant for any* $x \in U(e)$.

Proof. Since the lemma is trivial for $c = 0$, we may assume $c \neq 0$. Since $\infty \notin U(e)$, we can write $U(e) = D(\alpha, \rho)$ with some $\alpha \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$. The assumption yields $-\frac{d}{c} = \gamma^{-1}(\infty) \notin U(e)$ and thus

$$|cx+d| = |c|\left|x - (-\frac{d}{c})\right| > |c|\rho \quad \text{for any } x \in U(e).$$

On the other hand, for any $x, y \in U(e)$ we have

$$|(cx+d) - (cy+d)| = |c||x-y| \leqslant |c|\rho,$$

which yields $|cx+d| = |cy+d|$. $\square$

Lemma 9.9. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(K)$ *and let* $e \in \mathcal{T}_1^o$ *satisfying* $\infty \notin U(e)$ *and* $\infty \notin U(\gamma\circ e)$. *Then we have*

$$\rho(\gamma\circ e) = \rho(e)|cx+d|^{-2}|\det(\gamma)| \quad \textit{for any } x \in U(e).$$

Proof. Since $\infty \notin U(e)$, we have $U(e) = D(x, \rho(e))$ for any $x \in U(e)$. Since $\infty \notin U(\gamma\circ e) = \gamma(U(e))$, Lemma 4.8 concludes the proof. $\square$

Lemma 9.10. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(K)$ *and let* $e \in \mathcal{T}_1^o$ *satisfying* $\infty \notin U(e)$ *and* $\infty \in U(\gamma\circ e)$. *Then we have*

$$c \neq 0, \quad \rho(\gamma\circ e) = \rho(e)|c|^2|\det(\gamma)|^{-1}.$$

Proof. By assumption, we have $-\frac{d}{c} = \gamma^{-1}(\infty) \in U(e)$. Thus $c \neq 0$ and

$$U(e) = \left\{ z \in K_\infty \;\middle|\; \left| z + \frac{d}{c} \right| \leqslant \rho(e) \right\}.$$

Then Lemma 4.8 yields

$$U(\gamma \circ e) = \left\{ z \in K_\infty \;\middle|\; \left| z - \frac{a}{c} \right| \geqslant |\det(\gamma)||c|^{-2}\rho(e)^{-1} \right\} \cup \{\infty\},$$

from which the lemma follows. □

Lemma 9.11. *Let $e \in \mathcal{T}_1^o$ be any edge satisfying $\infty \notin U(e)$ and let $r \in U(e)$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ satisfying $\gamma^{-1}(r) \neq \infty$ and any integer $0 \leqslant i \leqslant k-2$, we have*

$$\begin{aligned} &\int_{U(e)} (x-r)^i d\mu_c(x) \\ &\quad = \sum_{j=0}^{k-2-i} \binom{k-2-i}{j} \det(\gamma)^{2-k+i} c^j (c\gamma^{-1}(r)+d)^{k-2-2i-j} \\ &\qquad \cdot \int_{U(\gamma^{-1}\circ e)} (x-\gamma^{-1}(r))^{i+j} d\mu_c(x). \end{aligned}$$

Proof. Lemma 9.7 (1) yields

$$\int_{U(e)} (x-r)^i d\mu_c(x) = \int_{U(\gamma^{-1}\circ e)} \det(\gamma)^{2-k} \left(\frac{ax+b}{cx+d} - r \right)^i (cx+d)^{k-2} d\mu_c(x).$$

Then we have

$$\begin{aligned} &\left(\frac{ax+b}{cx+d} - r \right)^i (cx+d)^{k-2} \\ &= (ax+b-r(cx+d))^i (cx+d)^{k-2-i} \\ &= ((a-cr)x-(dr-b))^i (cx+d)^{k-2-i} \\ &= (a-cr)^i \left(x - \frac{dr-b}{-cr+a} \right)^i (c(x-\gamma^{-1}(r)) + (c\gamma^{-1}(r)+d))^{k-2-i} \\ &= (a-cr)^i (x-\gamma^{-1}(r))^i \sum_{j=0}^{k-2-i} \binom{k-2-i}{j} c^j (x-\gamma^{-1}(r))^j (c\gamma^{-1}(r)+d)^{k-2-i-j}. \end{aligned}$$

Using the equality

$$\begin{aligned} (a-cr)(d+c\gamma^{-1}(r)) &= (a-cr)\left(d + c\left(\frac{dr-b}{-cr+a} \right) \right) \\ &= ad - cdr + c(dr-b) = \det(\gamma), \end{aligned}$$

we obtain

$$\left(\frac{ax+b}{cx+d}-r\right)^i (cx+d)^{k-2}$$
$$=\sum_{j=0}^{k-2-i}\binom{k-2-i}{j}\det(\gamma)^i c^j (x-\gamma^{-1}(r))^{i+j}(c\gamma^{-1}(r)+d)^{k-2-2i-j},$$

from which the lemma follows. □

Lemma 9.12. *Let $e\in\mathcal{T}_1^o$ be any edge satisfying $\infty\notin U(e)$ and let $r\in U(e)$. For any $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma$ satisfying $\gamma^{-1}(r)=\infty$ and any integer $0\leqslant i\leqslant k-2$, we have*

$$\int_{U(e)}(x-r)^i d\mu_c(x)=\det(\gamma)^{2-k+i}(-c)^{-i}\int_{U(\gamma^{-1}\circ e)}(cx+d)^{k-2-i}d\mu_c(x).$$

Proof. The assumption yields $r=\gamma(\infty)=\frac{a}{c}$. By Lemma 9.7 (1), we have

$$\int_{U(e)}(x-r)^i d\mu_c(x)=\int_{U(\gamma^{-1}\circ e)}\det(\gamma)^{2-k}\left(\frac{ax+b}{cx+d}-\frac{a}{c}\right)^i(cx+d)^{k-2}d\mu_c(x).$$

Then the lemma follows from

$$\left(\frac{ax+b}{cx+d}-\frac{a}{c}\right)^i(cx+d)^{k-2}=(-c)^{-i}\det(\gamma)^i(cx+d)^{k-2-i}.$$

□

Lemma 9.13. *There exists $C>0$ such that for any $e\in\mathcal{T}_1^o$ with $\infty\notin U(e)$, any $r\in U(e)$ and any $0\leqslant i\leqslant k-2$, we have*

$$\left|\int_{U(e)}(x-r)^i d\mu_c(x)\right|<C\rho(e)^{i-\frac{k-2}{2}}.$$

Proof. From Lemma 9.7 (1), we see that if $c(e')=0$ for some $e'\in\Gamma e$, then we have $\int_{U(e)}(x-r)^i d\mu_c(x)=0$ and the estimate in the lemma follows. By Proposition 9.5, we can take $e'_1,\ldots,e'_m\in\mathcal{T}_1^o$ such that any $e\in\mathcal{T}_1^o$ satisfying $c(e)\neq 0$ is equivalent to some $\pm e'_i$ modulo Γ. Replacing e'_i with $-e'_i$ if necessary, we may assume $\infty\notin U(e'_i)$ for any $i=1,\ldots,m$.

Take any $e\in\mathcal{T}_1^o$ satisfying $c(e)\neq 0$ and $\infty\notin U(e)$. Then $e=\pm\gamma\circ e'_s$ with some $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma$ and $s=1,\ldots,m$.

First suppose $e=\gamma\circ e'_s$. Then neither $U(e'_s)$ nor $U(\gamma\circ e'_s)=U(e)$ contains ∞. For any $r\in U(e)$, we have $\gamma^{-1}(r)\in U(e'_s)$ and $\gamma^{-1}(r)\neq\infty$.

Applying Lemma 9.9 to $x = \gamma^{-1}(r) \in U(e'_s)$ gives

$$\rho(e) = \rho(e'_s)|c\gamma^{-1}(r) + d|^{-2}.$$

Moreover, since $\infty \notin U(e'_s)$ and $-\frac{d}{c} = \gamma^{-1}(\infty) \notin U(e'_s)$, we have

$$|c\gamma^{-1}(r) + d| = |c||\gamma^{-1}(r) - \gamma^{-1}(\infty)| > |c|\rho(e'_s).$$

Then Lemma 9.11 yields

$$\begin{aligned}
&\rho(e)^{\frac{k-2}{2}-i}\left|\int_{U(e)}(x-r)^i d\mu_c(x)\right| \\
&\qquad \leqslant \max_{0\leqslant j\leqslant k-2-i} \rho(e'_s)^{\frac{k-2}{2}-i}|c|^j|c\gamma^{-1}(r)+d|^{-j}\left|\int_{U(e'_s)}(x-\gamma^{-1}(r))^{i+j}d\mu_c(x)\right| \\
&\qquad \leqslant \max_{0\leqslant j\leqslant k-2-i} \rho(e'_s)^{\frac{k-2}{2}-i-j}\left|\int_{U(e'_s)}(x-\gamma^{-1}(r))^{i+j}d\mu_c(x)\right|.
\end{aligned}$$

Since $\gamma^{-1}(r) \in U(e'_s)$, we have

$$\left|\int_{U(e'_s)}(x-\gamma^{-1}(r))^{i+j}d\mu_c(x)\right| \leqslant \max_{0\leqslant l\leqslant i+j,\ z\in U(e'_s)} \left|c(e'_s)(X^lY^{i+j-l})z^l\right|.$$

Since $\infty \notin U(e'_s)$, the value on the right-hand side is bounded by a real number depending only on (k, the harmonic cocycle c and) e'_s. Thus there exists a constant $C' > 0$ which satisfies the estimate of the lemma for any $e \in \bigcup_{s=1}^m \Gamma e'_s$.

Next suppose $e = -\gamma \circ e'_s$. Then we have $\infty \notin U(e'_s)$ and $\infty \in U(\gamma \circ e'_s) = U(-e)$. Lemma 9.10 gives $c \neq 0$, $\rho(-e) = \rho(e'_s)|c|^2$ and $\rho(e) = q^{-1}\rho(-e)^{-1} = (q\rho(e'_s))^{-1}|c|^{-2}$.

Put $r' := \frac{a}{c} = \gamma(\infty) \in \gamma(U(-e'_s)) = U(e)$. Then Lemma 9.12 yields

$$\begin{aligned}
&\rho(e)^{\frac{k-2}{2}-i}\left|\int_{U(e)}(x-r')^i d\mu_c(x)\right| \\
&\qquad = (q\rho(e'_s))^{i-\frac{k-2}{2}}|c|^{2i+2-k}|c|^{-i}\left|\int_{U(\gamma^{-1}\circ e)}(cx+d)^{k-2-i}d\mu_c(x)\right| \\
&\qquad = (q\rho(e'_s))^{i-\frac{k-2}{2}}\left|\int_{U(-e'_s)}\left(x+\frac{d}{c}\right)^{k-2-i}d\mu_c(x)\right|.
\end{aligned}$$

By Definition 9.1 (2), this equals

$$(q\rho(e'_s))^{i-\frac{k-2}{2}}\left|\int_{U(e'_s)}\left(x+\frac{d}{c}\right)^{k-2-i}d\mu_c(x)\right|.$$

Since $-\frac{d}{c} = \gamma^{-1}(\infty) \in \gamma^{-1}(U(-e)) = U(e'_s)$, this is bounded by

$$C'' = \max_{0\leqslant i\leqslant k-2}\ \max_{s=1,\ldots,m}\ \max_{0\leqslant l\leqslant k-2-i,\ z\in U(e'_s)} (q\rho(e'_s))^{i-\frac{k-2}{2}}\left|c(e'_s)(X^lY^{k-2-i-l})z^l\right|.$$

Let $r \in U(e)$. Then $|r' - r| \leqslant \rho(e)$ and we obtain

$$\begin{aligned}\left|\int_{U(e)} (x-r)^i d\mu_c(x)\right| &= \left|\int_{U(e)} (x - r' + (r'-r))^i \, d\mu_c(x)\right| \\ &\leqslant \max_{0 \leqslant j \leqslant i} \left|\int_{U(e)} (x-r')^j d\mu_c(x)\right| |r'-r|^{i-j} \\ &\leqslant \max_{0 \leqslant j \leqslant i} C'' \rho(e)^{j - \frac{k-2}{2}} \rho(e)^{i-j} = C'' \rho(e)^{i - \frac{k-2}{2}}.\end{aligned}$$

Thus there exists a constant $C''' > 0$ which satisfies the estimate of the lemma for any $e \in \bigcup_{s=1}^m \Gamma(-e'_s)$. This concludes the proof of the lemma. □

9.3. **Integration of meromorphic functions with poles only at ∞.**

Definition 9.14. We denote by $\mathscr{A}_k$ the set of $\mathbb{C}_\infty$-valued functions f on $\mathbb{P}^1(K_\infty)$ which are locally meromorphic with poles only at ∞ of order at most $k-2$. The latter condition means that for any $a \in \mathbb{P}^1(K_\infty)$, there exists $\nu \in \mathbb{Z}$ satisfying

$$f|_{D(a,q^{-\nu})} \in \begin{cases} \mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu})) & (a \neq \infty), \\ x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu})) & (a = \infty), \end{cases}$$

where we write

$$D_{\mathbb{C}_\infty}(\infty, q^{-\nu}) = D'_{\mathbb{C}_\infty}(0, q^{\nu}) = \mathrm{Sp}\left(\mathbb{C}_\infty\left\langle \frac{1}{\pi_\infty^\nu x}\right\rangle\right).$$

Then we have $P_k \subseteq \mathscr{A}_k$.

For any $a \in K_\infty$ and $\nu \in \mathbb{Z}$, we consider an element f of $\mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu}))$ or $x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu}))$ as an element of $\mathscr{A}_k$ by extending f by zero outside these discs.

In the sequel, we extend the integration of polynomials with respect to μ_c to that of elements of $\mathscr{A}_k$, following [MTT, §11]. Let $\mathrm{CO}(\mathbb{P}^1(K_\infty))$ be the set of compact open subsets of $\mathbb{P}^1(K_\infty)$.

Lemma 9.15. *Let $e \in \mathcal{T}_1^o$. If $\infty \in U(e)$ and $0 \notin U(e)$, then we have $U(e) = D(\infty, q^{-\nu})$ with some $\nu \in \mathbb{Z}$.*

Proof. Write $U(e) = D'(a, q^\nu)$ with some $a \in K_\infty$ and $\nu \in \mathbb{Z}$. Since $0 \notin U(e)$, we have $|a| < q^\nu$ and Lemma 4.2 implies $D^\circ(a, q^\nu) = D^\circ(0, q^\nu)$. Thus we obtain $U(e) = D'(0, q^\nu) = D(\infty, q^{-\nu})$. □

Theorem 9.16. *Let Γ be an arithmetic subgroup of $GL_2(K)$. Let $k \geqslant 2$ be an integer and $c \in C_k^{\mathrm{har}}(\Gamma)$. Then there exists a unique map*

$$\mathrm{CO}(\mathbb{P}^1(K_\infty)) \times \mathscr{A}_k \to \mathbb{C}_\infty, \quad (U, f) \mapsto \int_U f(x) d\mu_c(x)$$

satisfying the following conditions:

(1) $\int_U f(x) d\mu_c(x)$ *is finitely additive in* U *and* $\mathbb{C}_\infty$*-linear in* f*. The former condition means that if* $U_1, \ldots, U_r \in \mathrm{CO}(\mathbb{P}^1(K_\infty))$ *satisfy* $U = \coprod_{i=1}^r U_i$*, then we have*

$$\int_U f(x) d\mu_c(x) = \sum_{i=1}^r \int_{U_i} f(x) d\mu_c(x).$$

(2) *For any* $0 \leqslant i \leqslant k-2$ *and any* $e \in \mathcal{T}_1^o$*, we have*

$$\int_{U(e)} x^i d\mu_c(x) = (-1)^i c(e)(X^{k-2-i} Y^i).$$

(3) *There exists a constant* $C > 0$ *satisfying the following conditions:*

(a) *For any* $e \in \mathcal{T}_1^o$ *with* $\infty \notin U(e)$*, any* $a \in U(e)$ *and any integer* $i \geqslant 0$*, we have*

$$\left| \int_{U(e)} (x-a)^i d\mu_c(x) \right| \leqslant C \rho(e)^{i - \frac{k-2}{2}}.$$

(b) *For any* $e \in \mathcal{T}_1^o$ *with* $\infty \in U(e)$ *and* $0 \notin U(e)$ *and any integer* $i \geqslant -(k-2)$*, we have*

$$\left| \int_{U(e)} \frac{1}{x^i} d\mu_c(x) \right| \leqslant C \rho(e)^{i + \frac{k-2}{2}}.$$

(4) *Let* $e \in \mathcal{T}_1^o$*.*

(a) *Suppose* $\infty \notin U(e)$*. Write* $U(e) = D(a, q^{-\nu})$ *with some* $a \in K_\infty$ *and* $\nu \in \mathbb{Z}$*. Let* $F(x) = \sum_{i \geqslant 0} c_i (x-a)^i \in \mathbb{C}_\infty \langle \frac{x-a}{\pi_\infty^\nu} \rangle$*. Then we have*

$$\int_{U(e)} F(x) d\mu_c(x) = \sum_{i \geqslant 0} c_i \int_{U(e)} (x-a)^i d\mu_c(x).$$

(b) *Suppose* $\infty \in U(e)$ *and* $0 \notin U(e)$*. Write* $U(e) = D(\infty, q^{-\nu})$ *with some* $\nu \in \mathbb{Z}$*. Let* $F(x) = \sum_{i \geqslant -(k-2)} \frac{c_i}{x^i} \in x^{k-2} \mathbb{C}_\infty \langle \frac{1}{\pi_\infty^\nu x} \rangle$*. Then we have*

$$\int_{U(e)} F(x) d\mu_c(x) = \sum_{i \geqslant -(k-2)} c_i \int_{U(e)} \frac{1}{x^i} d\mu_c(x).$$

Corollary 9.17. *Let $c, c_1, c_2 \in C_k^{\mathrm{har}}(\Gamma)$ and $\lambda \in \mathbb{C}_\infty$. Note that $c_1 + c_2, \lambda c \in C_k^{\mathrm{har}}(\Gamma)$. For any $U \in \mathrm{CO}(\mathbb{P}^1(K_\infty))$ and $f \in \mathscr{A}_k$, we have*

$$\int_U f d\mu_{c_1+c_2}(x) = \int_U f d\mu_{c_1}(x) + \int_U f d\mu_{c_2}(x),$$
$$\int_U f d\mu_{\lambda c}(x) = \lambda \int_U f d\mu_c(x).$$

Proof. The map

$$\mathrm{CO}(\mathbb{P}^1(K_\infty)) \times \mathscr{A}_k \to \mathbb{C}_\infty, \quad (U, f) \mapsto \int_U f d\mu_{c_1}(x) + \int_U f d\mu_{c_2}(x)$$

satisfies all conditions of Theorem 9.16 for $c_1 + c_2$. Thus the uniqueness assertion of Theorem 9.16 yields the first equality.

For the second one, the map

$$\mathrm{CO}(\mathbb{P}^1(K_\infty)) \times \mathscr{A}_k \to \mathbb{C}_\infty, \quad (U, f) \mapsto \lambda \int_U f d\mu_c(x)$$

satisfies all conditions of Theorem 9.16 for λc, with the constant $(1 + |\lambda|)C$ for the assertion (3). Thus we obtain the second equality similarly. □

Definition 9.18. Let $a \in K_\infty$ and $\nu \in \mathbb{Z}$. For any

$$F(x) = \sum_{i \geqslant 0} c_i (x-a)^i \in \mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu})) = \mathbb{C}_\infty \left\langle \frac{x-a}{\pi_\infty^\nu} \right\rangle,$$

we define

$$I(F, a, \nu) = \pi_\infty^{-(k-1)\nu} \sum_{i \geqslant k-1} c_i \pi_\infty^{i\nu} \mathcal{O}_{\mathbb{C}_\infty}.$$

On the other hand, for any

$$F(x) = \sum_{i \geqslant -(k-2)} \frac{c_i}{x^i} \in x^{k-2} \mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu})) = x^{k-2} \mathbb{C}_\infty \left\langle \frac{1}{\pi_\infty^\nu x} \right\rangle,$$

we define

$$I(F, \infty, \nu) = \pi_\infty^{-\nu} \sum_{i \geqslant 1} c_i \pi_\infty^{i\nu} \mathcal{O}_{\mathbb{C}_\infty}.$$

Since $\lim_{i \to \infty} c_i \pi_\infty^{i\nu} = 0$ in both cases, we see that $I(F, a, \nu)$ is a finitely generated $\mathcal{O}_{\mathbb{C}_\infty}$-submodule of $\mathbb{C}_\infty$ for any $a \in \mathbb{P}^1(K_\infty)$. Since $\mathcal{O}_{\mathbb{C}_\infty}$ is a Bézout domain, we can write $I(F, a, \nu) = \alpha \mathcal{O}_{\mathbb{C}_\infty}$ with some $\alpha \in \mathbb{C}_\infty$. Then we define

$$|I(F, a, \nu)| = |\alpha|.$$

Lemma 9.19. *Let $a, a' \in K_\infty$ and $\nu, \nu' \in \mathbb{Z}$. Let $F(x) \in \mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu}))$. Then*

$$D(a', q^{-\nu'}) \subseteq D(a, q^{-\nu}) \quad \Rightarrow \quad I(F, a', \nu') \subseteq I(F, a, \nu).$$

Proof. Write $F(x) = \sum_{i \geqslant 0} c_i (x-a)^i$. From the assumption $D(a', q^{-\nu'}) \subseteq D(a, q^{-\nu})$, we see that $\nu' \geqslant \nu$.

When $a = a'$, we have

$$I(F, a, \nu') = \sum_{i \geqslant k-1} c_i \pi_\infty^{\nu'(i-(k-1))} \mathcal{O}_{\mathbb{C}_\infty} \subseteq \sum_{i \geqslant k-1} c_i \pi_\infty^{\nu(i-(k-1))} \mathcal{O}_{\mathbb{C}_\infty} = I(F, a, \nu)$$

and the lemma holds for this case. Since we have $D(a', q^{-\nu'}) \subseteq D(a', q^{-\nu}) \subseteq D(a, q^{-\nu})$, we may assume $\nu = \nu'$.

In the ring $\mathcal{O}(D_{\mathbb{C}_\infty}(a', q^{-\nu}))$ we can write

$$F(x) = \sum_{i \geqslant 0} c'_i (x - a')^i = \sum_{i \geqslant 0} c_i (x-a)^i = \sum_{i \geqslant 0} c_i (x - a' + (a' - a))^i,$$

which yields

$$c'_j = \sum_{i \geqslant j} c_i \binom{i}{j} (a' - a)^{i-j}.$$

Since $a' - a \in \pi_\infty^\nu \mathcal{O}_{\mathbb{C}_\infty}$, we have

$$c'_j \pi_\infty^{j\nu} \in \sum_{i \geqslant j} c_i \pi_\infty^{j\nu} (a' - a)^{i-j} \mathcal{O}_{\mathbb{C}_\infty} \subseteq \sum_{i \geqslant j} c_i \pi_\infty^{i\nu} \mathcal{O}_{\mathbb{C}_\infty}$$

and thus we obtain $I(F, a', \nu) \subseteq I(F, a, \nu)$. □

Lemma 9.20. *Let $\nu, \nu' \in \mathbb{Z}$. Let $F(x) \in x^{k-2} \mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu}))$. Then*

$$D(\infty, q^{-\nu'}) \subseteq D(\infty, q^{-\nu}) \quad \Rightarrow \quad I(F, \infty, \nu') \subseteq I(F, \infty, \nu).$$

Proof. Write $F(x) = \sum_{i \geqslant -(k-2)} \frac{c_i}{x^i}$. Since the assumption implies $\nu' \geqslant \nu$, we have

$$I(F, \infty, \nu') = \sum_{i \geqslant 1} c_i \pi_\infty^{(i-1)\nu'} \mathcal{O}_{\mathbb{C}_\infty} \subseteq \sum_{i \geqslant 1} c_i \pi_\infty^{(i-1)\nu} \mathcal{O}_{\mathbb{C}_\infty} = I(F, \infty, \nu).$$

□

Lemma 9.21. *Let $a \in K_\infty$ and $\nu, \nu' \in \mathbb{Z}$ satisfying $\nu > -\nu'$ and $|a| \geqslant q^\nu$. Let $F(x) \in x^{k-2} \mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu}))$. Then*

$$D(a, q^{-\nu'}) \subseteq D(\infty, q^{-\nu}), \quad I(F, a, \nu') \subseteq \pi_\infty^{\nu + (k-1)(1-\nu')} I(F, \infty, \nu).$$

If $\nu' \geqslant 1 + |\nu|$, then we also have

$$I(F, a, \nu') \subseteq \pi_\infty^{-k|\nu|} I(F, \infty, \nu).$$

Proof. The first assertion follows from Lemma 4.14. For the second assertion, write $F(x) = \sum_{i\geqslant -(k-2)} \frac{c_i}{x^i}$. In the ring $\mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu'}))$ we can write

$$\begin{aligned} F(x) &= \sum_{i\geqslant -(k-2)} \frac{c_i}{(a+(x-a))^i} = \sum_{i\geqslant -(k-2)} \frac{c_i}{a^i}\left(1+\frac{x-a}{a}\right)^{-i} \\ &= \sum_{i\geqslant -(k-2)} \sum_{j\geqslant 0} \binom{-i}{j} \frac{c_i}{a^i}\left(\frac{x-a}{a}\right)^j \\ &= \sum_{j\geqslant 0} \sum_{i\geqslant -(k-2)} \binom{-i}{j} \frac{c_i}{a^{i+j}}(x-a)^j. \end{aligned}$$

Since $\binom{-i}{j} = 0$ when $0 \leqslant -i < j$, this yields

$$\begin{aligned} I(F, a, \nu') &= \pi_\infty^{-(k-1)\nu'} \sum_{j\geqslant k-1} \sum_{i\geqslant -(k-2)} \binom{-i}{j} \frac{c_i}{a^{i+j}} \pi_\infty^{j\nu'} \mathcal{O}_{\mathbb{C}_\infty} \\ &= \pi_\infty^{-(k-1)\nu'} \sum_{j\geqslant k-1} \sum_{i\geqslant 1} \binom{-i}{j} \frac{c_i}{a^{i+j}} \pi_\infty^{j\nu'} \mathcal{O}_{\mathbb{C}_\infty} \\ &\subseteq \sum_{j\geqslant k-1} \sum_{i\geqslant 1} c_i \pi_\infty^{(i+j)\nu+(j-(k-1))\nu'} \mathcal{O}_{\mathbb{C}_\infty}. \end{aligned}$$

Since $\nu + \nu' \geqslant 1$, for any $j \geqslant k-1$ and $i \geqslant 1$ we have

$$\begin{aligned} (i+j)\nu + (j-(k-1))\nu' &= i\nu + j(\nu+\nu') - (k-1)\nu' \\ &\geqslant i\nu + (k-1) - (k-1)\nu' \\ &= i\nu + (k-1)(1-\nu'). \end{aligned}$$

Thus we obtain

$$I(F, a, \nu') \subseteq \sum_{i\geqslant 1} c_i \pi_\infty^{i\nu+(k-1)(1-\nu')} \mathcal{O}_{\mathbb{C}_\infty} = \pi_\infty^{\nu+(k-1)(1-\nu')} I(F, \infty, \nu).$$

Suppose $\nu' \geqslant 1 + |\nu|$. Since $\nu' - |\nu| > 0$ and $|\nu| + \nu \geqslant 0$, we have

$$\begin{aligned} &(i+j)\nu + (j-(k-1))\nu' \\ &\quad = (i-1)\nu + (\nu' - |\nu|)(j-(k-1)) + (j+1)(|\nu|+\nu) - k|\nu| \\ &\quad \geqslant (i-1)\nu - k|\nu|. \end{aligned}$$

Hence we obtain

$$I(F, a, \nu') \subseteq \sum_{i\geqslant 1} c_i \pi_\infty^{(i-1)\nu - k|\nu|} \mathcal{O}_{\mathbb{C}_\infty} = \pi_\infty^{-k|\nu|} I(F, \infty, \nu).$$

This concludes the proof. □

Lemma 9.22. *The map* $\mathrm{CO}(\mathbb{P}^1(K_\infty)) \times \mathscr{A}_k \to \mathbb{C}_\infty$ *satisfying all conditions of Theorem 9.16 is unique.*

Proof. If $(U,f)\mapsto \int_U f(x)d\mu_c(x)$ and $(U,f)\mapsto \int_U f(x)d\mu'_c(x)$ are two maps satisfying the conditions of Theorem 9.16, then the map

$$(U,f)\mapsto \int_U f(x)d\mu_c(x)-\int_U f(x)d\mu'_c(x)$$

also satisfies the conditions for $c=0$. Thus it is enough to show that for $c=0$, the conditions of the theorem imply $\int_U f(x)d\mu_c(x)=0$. By Lemma 4.15, we may assume $U=U(e)$ for some $e\in\mathcal{T}_1^o$, and we may also assume if $0\notin U(e)$ when $\infty\in U(e)$.

First suppose $\infty\notin U(e)$. Write $U(e)=D(a,q^{-\nu})$ with some $a\in K_\infty$ and $\nu\in\mathbb{Z}$. Take any $f(x)\in\mathcal{O}(D_{\mathbb{C}_\infty}(a,q^{-\nu}))$. For any $\nu'\geqslant\nu$, decompose $U(e)$ into a finite disjoint union of distinguished closed discs as

$$U(e)=\coprod_{a'\in\Lambda}D(a',\nu'),\quad D(a',\nu')=U(e_{a'}).$$

Write

$$f(x)=\sum_{i\geqslant 0}c_i(x-a')^i\in\mathbb{C}_\infty\left\langle\frac{x-a'}{\pi_\infty^{\nu'}}\right\rangle.$$

Using the assumption $c=0$, the conditions (2) and (4) give

$$\int_{U(e_{a'})}f(x)d\mu_0(x)=\sum_{i\geqslant k-1}c_i\int_{U(e_{a'})}(x-a')^i d\mu_0(x).$$

Then $\rho(e_{a'})=|\pi_\infty^{\nu'}|$ and the condition (3) yield

$$\left|\int_{U(e_{a'})}f(x)d\mu_0(x)\right|\leqslant C\left|\sum_{i\geqslant k-1}c_i\pi_\infty^{\nu'(i-\frac{k-2}{2})}\mathcal{O}_{\mathbb{C}_\infty}\right|=C|\pi_\infty|^{\frac{k\nu'}{2}}|I(f,a',\nu')|$$

and Lemma 9.19 shows

$$\left|\int_{U(e)}f(x)d\mu_0(x)\right|\leqslant\max_{a'\in\Lambda}\left|\int_{U(e_{a'})}f(x)d\mu_0(x)\right|\leqslant C|\pi_\infty|^{\frac{k\nu'}{2}}|I(f,a,\nu)|.$$

Since $k\geqslant 2$ and ν' is arbitrary, we obtain $\int_{U(e)}f(x)d\mu_0(x)=0$.

Next suppose $\infty\in U(e)$ and $0\notin U(e)$. By Lemma 9.15, we can write $U(e)=D(\infty,q^{-\nu})$ with some $\nu\in\mathbb{Z}$. Take any $f(x)\in x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty,q^{-\nu}))$. For any $\nu'\geqslant 1+|\nu|$, Lemma 4.14 implies that $U(e)$ is decomposed into a finite disjoint union of distinguished closed discs as

$$U(e)=D(\infty,q^{-\nu'})\sqcup\coprod_{a\in\Lambda}D(a,q^{-\nu'}),\quad D(a,q^{-\nu'})=U(e_a)$$

with some finite subset $\Lambda\subseteq K_\infty$.

Write $f(x) = \sum_{i \geqslant -(k-2)} \frac{c_i}{x^i}$. On $D(\infty, q^{-\nu'}) =: U(e')$, the conditions (2) for $c = 0$ and (4) yield

$$\int_{U(e')} f(x) d\mu_0(x) = \sum_{i \geqslant 1} c_i \int_{U(e')} \frac{1}{x^i} d\mu_0(x).$$

Since $\rho(e') = q^{-\nu'}$, the condition (3) implies

$$\left| \int_{U(e')} f(x) d\mu_0(x) \right| \leqslant C \left| \sum_{i \geqslant 1} c_i \pi_\infty^{\nu'(i + \frac{k-2}{2})} \mathcal{O}_{\mathbb{C}_\infty} \right| = C |\pi_\infty|^{\frac{k\nu'}{2}} |I(f, \infty, \nu')|.$$

By Lemma 9.20, this yields

$$\left| \int_{U(e')} f(x) d\mu_0(x) \right| \leqslant C |\pi_\infty|^{\frac{k\nu'}{2}} |I(f, \infty, \nu)|.$$

On $D(a, q^{-\nu'}) = U(e_a)$, the first part of the proof and Lemma 9.21 show

$$\left| \int_{U(e_a)} f(x) d\mu_0(x) \right| \leqslant C |\pi_\infty|^{\frac{k\nu'}{2}} |I(f, a', \nu')| \leqslant C |\pi_\infty|^{\frac{k(\nu' - 2|\nu|)}{2}} |I(f, \infty, \nu)|.$$

Since $k \geqslant 2$ and $\nu' \geqslant 1 + |\nu|$ is arbitrary, again we obtain $\int_{U(e)} f(x) d\mu_0(x) = 0$. $\square$

9.4. Construction of integration away from ∞.

Definition 9.23. Let $a \in K_\infty$ and $\nu \in \mathbb{Z}$. Let $f(x) = \sum_{i \geqslant 0} c_i (x-a)^i$ be an element of $\mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu}))$. Define

$$T_{a,\nu}(f) := \sum_{i=0}^{k-2} c_i (x-a)^i \in P_k.$$

Lemma 9.24. *Let $a, a' \in K_\infty$ and $\nu, \nu' \in \mathbb{Z}$ satisfying $D(a', q^{-\nu'}) \subseteq D(a, q^{-\nu})$. Let $f(x) = \sum_{i \geqslant 0} c_i (x-a)^i$ be an element of $\mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu}))$. Write*

$$T_{a,\nu}(f) - T_{a',\nu'}(f) = \sum_{i=0}^{k-2} b_i (x-a')^i.$$

Then we have

$$\pi_\infty^{i\nu} b_i \in \pi_\infty^{(k-1)\nu} I(f, a, \nu).$$

Proof. From the equality

$$\sum_{j \geqslant 0} c_j (x-a)^j = \sum_{j \geqslant 0} c_j (x - a' + (a' - a))^j = \sum_{j \geqslant 0} \sum_{j \geqslant i \geqslant 0} c_j \binom{j}{i} (a'-a)^{j-i} (x-a')^i,$$

we see that

$$T_{a',\nu'}(f)=\sum_{i=0}^{k-2}\left(\sum_{j\geqslant i}c_j\binom{j}{i}(a'-a)^{j-i}\right)(x-a')^i.$$

On the other hand, we have

$$\sum_{j=0}^{k-2}c_j(x-a)^j=\sum_{j=0}^{k-2}c_j(x-a'+(a'-a))^j=\sum_{j=0}^{k-2}\sum_{j\geqslant i\geqslant 0}c_j\binom{j}{i}(a'-a)^{j-i}(x-a')^i$$

which yields

$$T_{a,\nu}(f)=\sum_{i=0}^{k-2}\left(\sum_{k-2\geqslant j\geqslant i}c_j\binom{j}{i}(a'-a)^{j-i}\right)(x-a')^i.$$

Hence we obtain

$$\begin{aligned}\pi_\infty^{i\nu}b_i&=\pi_\infty^{i\nu}\sum_{j\geqslant k-1}c_j\binom{j}{i}(a'-a)^{j-i}\\&\in\sum_{j\geqslant k-1}c_j\pi_\infty^{(j-i)\nu+i\nu}\mathcal{O}_{\mathbb{C}_\infty}=\pi_\infty^{(k-1)\nu}I(f,a,\nu).\end{aligned}$$

□

Definition 9.25. Let $a\in K_\infty$ and $\nu\in\mathbb{Z}$. Let $f\in\mathcal{O}(D_{\mathbb{C}_\infty}(a,q^{-\nu}))$. For any $\nu'\geqslant\nu$, take any decomposition

$$D(a,q^{-\nu})=\coprod_{a'\in\Lambda_{\nu'}}D(a',q^{-\nu'}),\quad D(a',q^{-\nu'})=U(e_{a',\nu'})\tag{9.4}$$

with some finite subset $\Lambda_{\nu'}\subseteq K_\infty$. Then we define

$$m_{a,\nu,\nu'}(f)=\sum_{a'\in\Lambda_{\nu'}}\int_{U(e_{a',\nu'})}T_{a',\nu'}(f)d\mu_c(x),$$

where the integration on the right-hand side is given by (9.2).

Lemma 9.26. *Let $C>0$ be the constant in Lemma 9.13. Let $a\in K_\infty$ and $\nu\in\mathbb{Z}$. Let $f\in\mathcal{O}(D_{\mathbb{C}_\infty}(a,q^{-\nu}))$. Let ν' be an integer satisfying $\nu'\geqslant\nu$. For any $a'\in D(a,q^{-\nu'})$, we have $D(a,q^{-\nu'})=D(a',q^{-\nu'})$ and*

$$\left|\int_{D(a,q^{-\nu'})}(T_{a,\nu'}(f)-T_{a',\nu'}(f))d\mu_c(x)\right|\leqslant C|\pi_\infty|^{\frac{k}{2}\nu'}|I(f,a,\nu)|.$$

Proof. The first assertion follows from Lemma 4.2. By Lemma 9.13 and Lemma 9.24, we have

$$\begin{aligned}\left|\int_{D(a,q^{-\nu'})}(T_{a,\nu'}(f)-T_{a',\nu'}(f))d\mu_c(x)\right|&\\ \leqslant C\max_{i=0,\dots,k-2}|\pi_\infty|^{(k-1-i)\nu'}|I(f,a,\nu')||\pi_\infty|^{(i-\frac{k-2}{2})\nu'}&\\ = C|\pi_\infty|^{\frac{k}{2}\nu'}|I(f,a,\nu')|\leqslant C|\pi_\infty|^{\frac{k}{2}\nu'}|I(f,a,\nu)|,&\end{aligned}$$

where the last inequality follows from Lemma 9.19. □

Lemma 9.27. *Let $C>0$ be the constant in Lemma 9.13. Let $a\in K_\infty$ and $\nu\in\mathbb{Z}$. Let $f\in\mathcal{O}(D_{\mathbb{C}_\infty}(a,q^{-\nu}))$. Let ν',ν'' be integers satisfying $\nu''\geqslant\nu'\geqslant\nu$. Let $a'\in D(a,q^{-\nu})$ so that*

$$U(e_{a',\nu'}):=D(a',q^{-\nu'})\subseteq D(a,q^{-\nu}).$$

Take any decomposition

$$D(a',q^{-\nu'})=\coprod_{a''\in\Lambda}D(a'',q^{-\nu''}),\quad D(a'',q^{-\nu''})=:U(e_{a'',\nu''}).$$

Put

$$J_{a',\nu',\nu''}(f)=\sum_{a''\in\Lambda}\int_{U(e_{a'',\nu''})}(T_{a',\nu'}(f)-T_{a'',\nu''}(f))d\mu_c(x).$$

Then we have

$$|J_{a',\nu',\nu''}(f)|\leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a,\nu)|.$$

Proof. We claim that it is enough to show the lemma holds for $\nu''=\nu'+1$ for any $\nu'\geqslant\nu$ and $a'\in D(a,q^{-\nu})$. Indeed, since the case $\nu''=\nu'$ follows from Lemma 9.26, by induction we may assume that the lemma holds for some $\nu''\geqslant\nu'$. Take any decompositions

$$D(a',q^{-\nu'})=\coprod_{a''\in\Lambda}D(a'',q^{-\nu''})=\coprod_{b\in\Lambda'}D(b,q^{-\nu''-1}).$$

From Lemma 4.2, we see that the latter is a refinement of the former. For any $a''\in\Lambda$, we can find a subset $\Lambda(a'')\subseteq\Lambda'$ satisfying

$$D(a'',q^{-\nu''})=\coprod_{b\in\Lambda(a'')}D(b,q^{-\nu''-1}),\quad D(b,q^{-\nu''-1})=U(e_{b,\nu''+1}).$$

Then Lemma 9.6 yields

$$c(e_{a'',\nu''})=\sum_{b\in\Lambda(a'')}c(e_{b,\nu''+1}).$$

and also

$$
\begin{aligned}
J_{a',\nu',\nu''+1}(f) &= \sum_{a''\in\Lambda}\sum_{b\in\Lambda(a'')}\int_{U(e_{b,\nu''+1})}(T_{a',\nu'}(f)-T_{b,\nu''+1}(f))d\mu_c(x)\\
&= \sum_{a''\in\Lambda}\sum_{b\in\Lambda(a'')}\int_{U(e_{b,\nu''+1})}(T_{a',\nu'}(f)-T_{a'',\nu''}(f))d\mu_c(x)\\
&\quad+\sum_{a''\in\Lambda}\sum_{b\in\Lambda(a'')}\int_{U(e_{b,\nu''+1})}(T_{a'',\nu''}(f)-T_{b,\nu''+1}(f))d\mu_c(x)\\
&= \sum_{a''\in\Lambda}\int_{U(e_{a'',\nu''})}(T_{a',\nu'}(f)-T_{a'',\nu''}(f))d\mu_c(x)\\
&\quad+\sum_{a''\in\Lambda}\sum_{b\in\Lambda(a'')}\int_{U(e_{b,\nu''+1})}(T_{a'',\nu''}(f)-T_{b,\nu''+1}(f))d\mu_c(x)\\
&= J_{a',\nu',\nu''}(f)+\sum_{a''\in\Lambda}J_{a'',\nu'',\nu''+1}(f).
\end{aligned}
$$

On the other hand, the assumptions yield

$$
\begin{aligned}
|J_{a',\nu',\nu''}(f)| &\leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a,\nu)|,\\
|J_{a'',\nu'',\nu''+1}(f)| &\leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu''}|I(f,a,\nu)|.
\end{aligned}
$$

Since $k>0$, we obtain

$$
\begin{aligned}
|J_{a',\nu',\nu''+1}(f)| &\leqslant C\max\{|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'},|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu''}\}|I(f,a,\nu)|\\
&= C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a,\nu)|
\end{aligned}
$$

and the claim follows.

Now we assume $\nu''=\nu'+1$. Write

$$
T_{a',\nu'}(f)-T_{a'',\nu'+1}(f)=\sum_{i=0}^{k-2}b_i(x-a'')^i.
$$

By Lemma 9.13 and Lemma 9.24, we have

$$\begin{aligned}
|J_{a',\nu',\nu'+1}(f)| &\leqslant \max_{a''\in\Lambda}\left|\int_{U(e_{a'',\nu'+1})}(T_{a',\nu'}(f)-T_{a'',\nu'+1}(f))\,d\mu_c(x)\right|\\
&\leqslant \max_{a''\in\Lambda,\ 0\leqslant i\leqslant k-2}\left|b_i\int_{U(e_{a'',\nu'+1})}(x-a'')^i d\mu_c(x)\right|\\
&\leqslant \max_{a''\in\Lambda,\ 0\leqslant i\leqslant k-2} C|\pi_\infty|^{(\nu'+1)(i-\frac{k-2}{2})}\left|\pi_\infty^{(k-1-i)\nu'}I(f,a',\nu')\right|\\
&= \max_{0\leqslant i\leqslant k-2} C|\pi_\infty|^{(\nu'+1)(i-\frac{k-2}{2})}\left|\pi_\infty^{(k-1-i)\nu'}I(f,a',\nu')\right|\\
&= \max_{0\leqslant i\leqslant k-2} C|\pi_\infty|^{i-\frac{k-2}{2}(\nu'+1)+(k-1)\nu'}|I(f,a',\nu')|\\
&= C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a',\nu')|.
\end{aligned}$$

Now Lemma 9.19 yields

$$|J_{a',\nu',\nu'+1}(f)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a,\nu)|.$$

This concludes the proof. □

Lemma 9.28. *Let $C>0$ be the constant in Lemma 9.13. Then for any $\nu'\geqslant\nu$ we have*

$$|m_{a,\nu,\nu'+1}(f)-m_{a,\nu,\nu'}(f)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a,\nu)|.$$

Proof. We have two coverings

$$D(a,q^{-\nu}) = \coprod_{a'\in\Lambda_{\nu'}} D(a',q^{-\nu'}) = \coprod_{a''\in\Lambda_{\nu'+1}} D(a'',q^{-\nu'-1}).$$

By Lemma 4.2, this forces the latter to be a refinement of the former.

For any $a'\in\Lambda_{\nu'}$, take a subset $\Lambda(a')\subseteq\Lambda_{\nu'+1}$ satisfying

$$D(a',q^{-\nu'}) = \coprod_{a''\in\Lambda(a')} D(a'',q^{-\nu'-1}).$$

Then Lemma 9.6 yields

$$\begin{aligned}
&m_{a,\nu,\nu'}(f)-m_{a,\nu,\nu'+1}(f) =\\
&\quad\sum_{a'\in\Lambda_{\nu'}}\int_{U(e_{a',\nu'})}T_{a',\nu'}(f)d\mu_c(x) - \sum_{a'\in\Lambda_{\nu'}}\sum_{a''\in\Lambda(a')}\int_{U(e_{a'',\nu'+1})}T_{a'',\nu'+1}(f)d\mu_c(x)\\
&\quad= \sum_{a'\in\Lambda_{\nu'}}\sum_{a''\in\Lambda(a')}\int_{U(e_{a'',\nu'+1})}(T_{a',\nu'}(f)-T_{a'',\nu'+1}(f))d\mu_c(x)\\
&\quad= \sum_{a'\in\Lambda_{\nu'}} J_{a',\nu',\nu'+1}(f).
\end{aligned}$$

Then the lemma follows from Lemma 9.27. □

Lemma 9.29. *The sequence* $\{m_{a,\nu,\nu'}(f)\}_{\nu'\geqslant\nu}$ *converges in* $\mathbb{C}_\infty$.

Proof. By Lemma 6.6, the sequence $\{m_{a,\nu,\nu'}(f)\}_{\nu'\geqslant\nu}$ is Cauchy if and only if

$$\lim_{\nu'\to\infty}|m_{a,\nu,\nu'+1}(f)-m_{a,\nu,\nu'}(f)|=0.$$

Since $k\geqslant 2$, this follows from Lemma 9.28. □

Lemma 9.30. *The limit* $\lim_{\nu'\to\infty} m_{a,\nu,\nu'}(f)$ *is independent of the choice of* $\Lambda_{\nu'}$ *chosen to define each* $m_{a,\nu,\nu'}(f)$.

Proof. For any $\nu'\geqslant\nu$, write $\Lambda_{\nu'}=\{a'_1,\ldots,a'_r\}$. Take any $\tilde{a}'_l\in D(a'_l,q^{-\nu'})$ and put $\tilde{\Lambda}_{\nu'}=\{\tilde{a}'_1,\ldots,\tilde{a}'_r\}$. Then $D(a'_l,q^{-\nu'})=D(\tilde{a}'_l,q^{-\nu'})$. Put

$$\tilde{m}_{a,\nu,\nu'}(f)=\sum_{l=1}^{r}\int_{D(a'_l,q^{-\nu'})}T_{\tilde{a}'_l,\nu'}(f)d\mu_c(x).$$

By Lemma 9.26, we have

$$\begin{aligned}|\tilde{m}_{a,\nu,\nu'}(f)-m_{a,\nu,\nu'}(f)|&=\left|\sum_{l=1}^{r}\int_{D(a'_l,q^{-\nu'})}(T_{\tilde{a}'_l,\nu'}(f)-T_{a'_l,\nu'}(f))d\mu_c(x)\right|\\&\leqslant C|\pi_\infty|^{\frac{k}{2}\nu'}|I(f,a,\nu)|.\end{aligned}$$

Since $k\geqslant 2$, we have $\lim_{\nu'\to\infty}|\tilde{m}_{a,\nu,\nu'}(f)-m_{a,\nu,\nu'}(f)|=0$, which implies the lemma. □

Definition 9.31. Let $e\in\mathcal{T}_1^o$ satisfying $U(e)=D(a,q^{-\nu})$ with some $a\in K_\infty$ and $\nu\in\mathbb{Z}$. Let $f\in\mathcal{O}(D_{\mathbb{C}_\infty}(a,q^{-\nu}))$. We define

$$\int_{U(e)}f(x)d\mu_c(x):=\lim_{\nu'\to\infty}m_{a,\nu,\nu'}(f).$$

9.5. Construction of integration around ∞.

Definition 9.32. Let $\nu\in\mathbb{Z}$ and let $f(x)=\sum_{i\geqslant-(k-2)}\frac{c_i}{x^i}$ be an element of $x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty,q^{-\nu}))$. Define

$$T_{\infty,\nu}(f):=\sum_{i=-(k-2)}^{0}\frac{c_i}{x^i}\in P_k.$$

Note that for any $\nu'\geqslant\nu$, we have $T_{\infty,\nu}(f)=T_{\infty,\nu'}(f)$.

Lemma 9.33. *Let* $a'\in K_\infty$ *and* $\nu,\nu'\in\mathbb{Z}$ *satisfying* $\nu>-\nu'$ *and* $|a'|\geqslant q^\nu$ *so that Lemma 4.14 implies* $D(a',q^{-\nu'})\subseteq D(\infty,q^{-\nu})$. *Let*

$f(x) = \sum_{i \geqslant -(k-2)} \frac{c_i}{x^i}$ be an element of $x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu}))$. Write

$$T_{\infty,\nu}(f) - T_{a',\nu'}(f) = \sum_{i=0}^{k-2} b_i (x-a')^i.$$

Then we have

$$\pi_\infty^{-i\nu} b_i \in \pi_\infty^{\nu} I(f, \infty, \nu).$$

Proof. From the equality

$$\sum_{j \geqslant -(k-2)} \frac{c_j}{x^j} = \sum_{j \geqslant -(k-2)} \frac{c_j}{(a' + (x-a'))^j} = \sum_{i \geqslant 0} \sum_{j \geqslant -(k-2)} \binom{-j}{i} \frac{c_j}{(a')^{i+j}} (x-a')^i,$$

we see that

$$T_{a',\nu'}(f) = \sum_{i=0}^{k-2} \sum_{j \geqslant -(k-2)} \binom{-j}{i} \frac{c_j}{(a')^{i+j}} (x-a')^i.$$

On the other hand, since $\binom{-j}{i} = 0$ when $0 \leqslant -j < i$, we have

$$\begin{aligned} \sum_{j=-(k-2)}^{0} \frac{c_j}{x^j} = \sum_{j=-(k-2)}^{0} \frac{c_j}{(a' + (x-a'))^j} &= \sum_{i \geqslant 0} \sum_{j=-(k-2)}^{0} \binom{-j}{i} \frac{c_j}{(a')^{i+j}} (x-a')^i \\ &= \sum_{i=0}^{k-2} \sum_{j=-(k-2)}^{0} \binom{-j}{i} \frac{c_j}{(a')^{i+j}} (x-a')^i, \end{aligned}$$

which yields

$$b_i = -\sum_{j \geqslant 1} \binom{-j}{i} \frac{c_j}{(a')^{i+j}}.$$

Since $|a'|^{-1} \leqslant q^{-\nu} = |\pi_\infty|^\nu$, we obtain

$$\pi_\infty^{-i\nu} b_i \in \pi_\infty^{-i\nu} \sum_{j \geqslant 1} c_j \pi_\infty^{(i+j)\nu} \mathcal{O}_{\mathbb{C}_\infty} = \sum_{j \geqslant 1} c_j \pi_\infty^{j\nu} \mathcal{O}_{\mathbb{C}_\infty} \in \pi_\infty^{\nu} I(f, \infty, \nu).$$

□

Definition 9.34. Let $\nu \in \mathbb{Z}$. Let $f \in x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu}))$. For any $\nu' \geqslant 1 + |\nu|$, by Lemma 4.14 we can take a decomposition

$$D(\infty, q^{-\nu}) = D(\infty, q^{-\nu'}) \sqcup \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'})$$

with some finite subset $\Lambda_{\nu'} \subseteq K_\infty$. Write $D(a', q^{-\nu'}) = U(e_{a',\nu'})$ for any $a' \in \Lambda_{\nu'} \cup \{\infty\}$. Then we define

$$m_{\infty,\nu,\nu'}(f) = \int_{U(e_{\infty,\nu'})} T_{\infty,\nu'}(f) d\mu_c(x) + \sum_{a' \in \Lambda_{\nu'}} \int_{U(e_{a',\nu'})} T_{a',\nu'}(f) d\mu_c(x),$$

where the integration on the right-hand side is given by (9.2).

Lemma 9.35. *Let $\nu', \tilde{\nu} \in \mathbb{Z}$ satisfying $\nu' > -\tilde{\nu}$. Let $a' \in K_\infty$ satisfying $|a'| \geqslant q^{\nu'}$ so that $D(a', q^{-\tilde{\nu}}) \subseteq D(\infty, q^{-\nu'})$ by Lemma 4.14. Let $f \in x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu'}))$. Take a decomposition*

$$D(a', q^{-\tilde{\nu}}) = \coprod_{\tilde{a} \in \Lambda_{\tilde{\nu}+1}(a')} D(\tilde{a}, q^{-\tilde{\nu}-1}).$$

Put

$$J_{a',\tilde{\nu},\tilde{\nu}+1}(f) := \int_{D(a',q^{-\tilde{\nu}})} T_{a',\tilde{\nu}}(f) d\mu_c(x) - \sum_{\tilde{a} \in \Lambda_{\tilde{\nu}+1}(a')} \int_{D(\tilde{a},q^{-\tilde{\nu}-1})} T_{\tilde{a},\tilde{\nu}+1}(f) d\mu_c(x).$$

Then we have

$$|J_{a',\tilde{\nu},\tilde{\nu}+1}(f)| \leqslant C|\pi_\infty|^{1+\frac{k}{2}\nu'} |I(f, \infty, \nu')|,$$

where C is the constant in Lemma 9.13.

Proof. By Lemma 9.6 and (9.2), we have

$$J_{a',\tilde{\nu},\tilde{\nu}+1}(f) = \sum_{\tilde{a} \in \Lambda_{\tilde{\nu}+1}(a')} \int_{D(\tilde{a},q^{-\tilde{\nu}-1})} (T_{a',\tilde{\nu}}(f) - T_{\tilde{a},\tilde{\nu}+1}(f)) d\mu_c(x).$$

Write

$$T_{a',\tilde{\nu}}(f) - T_{\tilde{a},\tilde{\nu}+1}(f) = \sum_{i=0}^{k-2} b_i (x - \tilde{a})^i.$$

Then Lemma 9.13 and Lemma 9.24 yield

$$\begin{aligned}
|J_{a',\tilde{\nu},\tilde{\nu}+1}(f)| &\leqslant C \max_{i=0,\ldots,k-2} |\pi_\infty|^{(i-\frac{k-2}{2})(\tilde{\nu}+1)+(k-1-i)\tilde{\nu}} |I(f, a', \tilde{\nu})| \\
&= C \max_{i=0,\ldots,k-2} |\pi_\infty|^{i-\frac{k-2}{2}+\frac{k\tilde{\nu}}{2}} |I(f, a', \tilde{\nu})| \\
&= C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k\tilde{\nu}}{2}} |I(f, a', \tilde{\nu})|.
\end{aligned}$$

Since Lemma 4.14 implies

$$D(a', q^{-\tilde{\nu}}) \subseteq D(a', q^{\nu'-1}) \subseteq D(\infty, q^{-\nu'}),$$

by Lemma 9.19 and Lemma 9.21 we have

$$I(f, a', \tilde{\nu}) \subseteq I(f, a', 1-\nu') \subseteq \pi_\infty^{k\nu'} I(f, \infty, \nu').$$

Since $\tilde{\nu} \geqslant 1 - \nu'$ and $k \geqslant 2$, we obtain

$$\begin{aligned}
|J_{a',\tilde{\nu},\tilde{\nu}+1}(f)| &\leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}(1-\nu')+k\nu'} |I(f, \infty, \nu')| \\
&= C|\pi_\infty|^{1+\frac{k}{2}\nu'} |I(f, \infty, \nu')|.
\end{aligned}$$

This concludes the proof. $\square$

Lemma 9.36. *Let $C > 0$ be the constant in Lemma 9.13. Then for any $\nu' \geqslant 1 + |\nu|$ we have*

$$|m_{\infty,\nu,\nu'+1}(f) - m_{\infty,\nu,\nu'}(f)| \leqslant C|I(f,\infty,\nu)||\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'-k|\nu|}.$$

Proof. We have two coverings

$$D(\infty, q^{-\nu}) = \coprod_{a' \in \Lambda_{\nu'} \cup \{\infty\}} D(a', q^{-\nu'}) = \coprod_{a'' \in \Lambda_{1+\nu'} \cup \{\infty\}} D(a'', q^{-(1+\nu')}),$$

which yields

$$\{x \in K_\infty \mid |x| = q^{\nu'}\} \sqcup \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}) = \coprod_{a'' \in \Lambda_{1+\nu'}} D(a'', q^{-(1+\nu')}).$$

Since $\nu' > -(1+\nu')$, we have

$$|a''| = q^{\nu'} \quad \Rightarrow \quad D(a'', q^{-(1+\nu')}) \subseteq \{x \in K_\infty \mid |x| = q^{\nu'}\}.$$

This forces the latter covering to be a refinement of the former.

For any $a' \in \Lambda_{\nu'} \cup \{\infty\}$, take a subset $\Lambda(a') \subseteq \Lambda_{1+\nu'} \cup \{\infty\}$ satisfying

$$D(a', q^{-\nu'}) = \coprod_{a'' \in \Lambda(a')} D(a'', q^{-(1+\nu')}).$$

Then Lemma 9.6 and (9.2) imply

$$\begin{aligned} J_{a',\nu'}(f) &:= \int_{U(e_{a',\nu'})} T_{a',\nu'}(f) d\mu_c(x) - \sum_{a'' \in \Lambda(a')} \int_{U(e_{a'',1+\nu'})} T_{a'',1+\nu'}(f) d\mu_c(x) \\ &= \sum_{a'' \in \Lambda(a')} \int_{U(e_{a'',1+\nu'})} \left(T_{a',\nu'}(f) - T_{a'',1+\nu'}(f)\right) d\mu_c(x). \end{aligned}$$

Note that we have

$$m_{\infty,\nu,\nu'}(f) - m_{\infty,\nu,\nu'+1}(f) = \sum_{a' \in \Lambda_{\nu'} \cup \{\infty\}} J_{a',\nu'}(f).$$

If $a'' = \infty$ and $a'' \in \Lambda(a')$, then we also have $a' = \infty$ and

$$T_{\infty,\nu'}(f) - T_{\infty,1+\nu'}(f) = 0.$$

Hence the term for $a'' = \infty$ has no contribution to $J_{a',\nu'}(f)$.

Suppose $a'' \neq \infty$. Write

$$T_{a',\nu'}(f) - T_{a'',1+\nu'}(f) = \sum_{i=0}^{k-2} b_i (x - a'')^i.$$

If $a' \neq \infty$, then as in the proof of Lemma 9.27, we have

$$\left|\int_{U(e_{a'',1+\nu'})} (T_{a',\nu'}(f) - T_{a'',1+\nu'}(f))\, d\mu_c(x)\right| \leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,a',\nu')|$$
$$\leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'-k|\nu|}|I(f,\infty,\nu)|,$$

where the last inequality follows from Lemma 9.21. Thus we obtain

$$|J_{a',\nu'}(f)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'-k|\nu|}|I(f,\infty,\nu)| \tag{9.5}$$

for any $a' \neq \infty$.

Let us consider the case $a' = \infty$. In this case, we have

$$\{x \in K_\infty \mid |x| = q^{\nu'}\} = \coprod_{a'' \in \Lambda(\infty)\backslash\{\infty\}} D(a'', q^{-(1+\nu')}).$$

For any integer $\tilde{\nu} \in [1-\nu', \nu'+1]$, we can find a subset $\Lambda_{\tilde{\nu}} \subseteq \Lambda(\infty)\backslash\{\infty\}$ satisfying $\Lambda_{\nu'+1} = \Lambda(\infty)\backslash\{\infty\}$, $\Lambda_{\tilde{\nu}-1} \subseteq \Lambda_{\tilde{\nu}}$ and

$$\{x \in K_\infty \mid |x| = q^{\nu'}\} = \coprod_{a'' \in \Lambda_{\tilde{\nu}}} D(a'', q^{-\tilde{\nu}}).$$

For each $\tilde{\nu}$ and any $a'' \in \Lambda_{\tilde{\nu}}$, we can write

$$D(a'', q^{-\tilde{\nu}}) = \coprod_{\tilde{a} \in \Lambda_{\tilde{\nu}+1}(a'')} D(\tilde{a}, q^{-\tilde{\nu}-1})$$

with some subset $\Lambda_{\tilde{\nu}+1}(a'') \subseteq \Lambda_{\tilde{\nu}+1}$. By applying Lemma 9.35 for any $\tilde{\nu} \in [1-\nu', \nu']$, we obtain

$$\begin{aligned} J_{\infty,\nu'}(f) &\in \int_{D(\infty,q^{-\nu'})} T_{\infty,\nu'}(f) d\mu_c(x) - \int_{D(\infty,q^{-(1+\nu')})} T_{\infty,1+\nu'}(f) d\mu_c(x) \\ &\quad - \sum_{a'' \in \Lambda_{1-\nu'}} \int_{D(a'',q^{\nu'-1})} T_{a'',1-\nu'}(f) d\mu_c(x) + I \\ &= \sum_{a'' \in \Lambda_{1-\nu'}} \int_{D(a'',q^{\nu'-1})} (T_{\infty,\nu'}(f) - T_{a'',1-\nu'}(f)) d\mu_c(x) + I \end{aligned}$$

with some monogenic $\mathcal{O}_{\mathbb{C}_\infty}$-submodule $I \subseteq \mathbb{C}_\infty$ satisfying

$$|I| \leqslant C|\pi_\infty|^{1+\frac{k}{2}\nu'}|I(f,\infty,\nu')| \leqslant C|\pi_\infty|^{1+\frac{k}{2}\nu'}|I(f,\infty,\nu)|, \tag{9.6}$$

where the last inequality follows from Lemma 9.20.

Finally, writing

$$T_{\infty,\nu'}(f) - T_{a'',1-\nu'}(f) = \sum_{i=0}^{k-2} b_i (x-a'')^i,$$

we see from Lemma 9.13 and Lemma 9.33 that

$$\begin{aligned}
&\left|\int_{D(a'',q^{\nu'-1})}(T_{\infty,\nu'}(f)-T_{a'',1-\nu'}(f))d\mu_c(x)\right| \\
&\leqslant \max_{0\leqslant i\leqslant k-2}\left|b_i\int_{D(a'',q^{\nu'-1})}(x-a'')^i d\mu_c(x)\right| \\
&\leqslant \max_{0\leqslant i\leqslant k-2} C|\pi_\infty|^{(1-\nu')(i-\frac{k-2}{2})}\left|\pi_\infty^{(i+1)\nu'}I(f,\infty,\nu')\right| \\
&= \max_{0\leqslant i\leqslant k-2} C|\pi_\infty|^{i-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,\infty,\nu')| \\
&= C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,\infty,\nu')| \\
&\leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,\infty,\nu)|,
\end{aligned}$$

where Lemma 9.20 gives the last inequality. Since $k \geqslant 2$, by the inequality (9.6) we obtain

$$|J_{\infty,\nu'}(f)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}\nu'}|I(f,\infty,\nu)|. \tag{9.7}$$

Now the lemma follows from (9.5) and (9.7). □

Lemma 9.37. *The sequence* $\{m_{\infty,\nu,\nu'}(f)\}_{\nu'\geqslant 1+|\nu|}$ *converges in* $\mathbb{C}_\infty$.

Proof. Since $k \geqslant 2$, Lemma 9.36 yields $\lim_{\nu'\to\infty}|m_{\infty,\nu,\nu'+1}(f)-m_{\infty,\nu,\nu'}(f)| = 0$. Thus Lemma 6.6 implies that $\{m_{\infty,\nu,\nu'}(f)\}_{\nu'\geqslant 1+|\nu|}$ converges. □

Lemma 9.38. *The limit* $\lim_{\nu'\to\infty} m_{\infty,\nu,\nu'}(f)$ *is independent of the choice of* $\Lambda_{\nu'}$ *chosen to define each* $m_{\infty,\nu,\nu'}(f)$.

Proof. For any $\nu' \geqslant 1+|\nu|$, write $\Lambda_{\nu'} = \{a'_1,\ldots,a'_r\}$. Take any $\tilde{a}'_l \in D(a'_l, q^{-\nu'})$ and put $\tilde{\Lambda}_{\nu'} = \{\tilde{a}'_1,\ldots,\tilde{a}'_r\}$. Then $U(e_{a'_l,\nu'}) = D(a'_l,q^{-\nu'}) = D(\tilde{a}'_l,q^{-\nu'})$. Put

$$\tilde{m}_{\infty,\nu,\nu'}(f) = \int_{U(e_{\infty,\nu'})}T_{\infty,\nu'}(f)d\mu_c(x)+\sum_{l=1}^{r}\int_{U(e_{a'_l,\nu'})}T_{\tilde{a}'_l,\nu'}(f)d\mu_c(x).$$

By Lemma 9.13 and Lemma 9.24 combined with Lemma 9.21, we have

$$\begin{aligned}
&|\tilde{m}_{\infty,\nu,\nu'}(f)-m_{\infty,\nu,\nu'}(f)| \\
&\qquad\leqslant \max_{l=1,\ldots,r}\left|\sum_{i=0}^{k-2}\int_{U(e_{a'_l,\nu'})}(x-a'_l)^i d\mu_c(x)\right||\pi_\infty^{(k-1-i)\nu'}I(f,a'_l,\nu')| \\
&\qquad\leqslant \max_{i=0,\ldots,k-2} C|\pi_\infty|^{\nu'(i-\frac{k-2}{2})}|\pi_\infty|^{(k-1-i)\nu'-k|\nu|}|I(f,\infty,\nu)| \\
&\qquad= C|\pi_\infty|^{\frac{k\nu'}{2}-k|\nu|}|I(f,\infty,\nu)|.
\end{aligned}$$

Since $k \geqslant 2$, we have $\lim_{\nu'\to\infty}|\tilde{m}_{\infty,\nu,\nu'}(f)-m_{\infty,\nu,\nu'}(f)| = 0$, which implies the lemma. □

Definition 9.39. Let $e \in \mathcal{T}_1^o$ satisfying $U(e) = D(\infty, q^{-\nu})$ with some $\nu \in \mathbb{Z}$. Let $f \in x^{k-2}\mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu}))$. We define

$$\int_{U(e)} f(x) d\mu_c(x) := \lim_{\nu' \to \infty} m_{\infty,\nu,\nu'}(f).$$

9.6. **Properties of integration.**

Definition 9.40. For any $U \in \mathrm{CO}(\mathbb{P}^1(K_\infty))$ and $f \in \mathscr{A}_k$, by Lemma 4.15 we choose a decomposition

$$U = \coprod_{a \in \Lambda} D(a, q^{-\nu_a}) \tag{9.8}$$

with some finite subset $\Lambda \subseteq \mathbb{P}^1(K_\infty)$ and put

$$\int_U f(x) d\mu_c(x) := \sum_{a \in \Lambda} \int_{D(a, q^{-\nu_a})} f(x) d\mu_c(x).$$

Lemma 9.41. *The integration $\int_U f(x) d\mu_c(x)$ is independent of the choice of a decomposition (9.8) of U.*

Proof. Take two decompositions of U as in (9.8). By Lemma 4.16, we may assume that one is a refinement of the other. Thus we may assume $U = D(b, q^{-\nu})$ with some $b \in \mathbb{P}^1(K_\infty)$ and $\nu \in \mathbb{Z}$.

Take any $\nu' \in \mathbb{Z}$ satisfying $\nu' \geqslant 1 + |\nu|$ and $\nu' \geqslant 1 + |\nu_a|$ for any $a \in \Lambda$. By Lemma 4.14, we can choose a decomposition

$$D(a, q^{-\nu_a}) = \coprod_{a' \in \Lambda_{a,\nu'}} D(a', q^{-\nu'})$$

with some finite subset $\Lambda_{a,\nu'} \subseteq \mathbb{P}^1(K_\infty)$. Then we have

$$D(b, q^{-\nu}) = \coprod_{a \in \Lambda} \coprod_{a' \in \Lambda_{a,\nu'}} D(a', q^{-\nu'})$$

and thus we may assume

$$m_{b,\nu,\nu'}(f) = \sum_{a \in \Lambda} m_{a,\nu_a,\nu'}(f).$$

Taking $\lim_{\nu' \to \infty}$ we obtain

$$\int_{D(b,q^{-\nu})} f(x) d\mu_c(x) = \sum_{a \in \Lambda} \int_{D(a,q^{-\nu_a})} f(x) d\mu_c(x).$$

This concludes the proof. $\square$

Lemma 9.42. *If $U_1, \ldots, U_r \in \mathrm{CO}(\mathbb{P}^1(K_\infty))$ satisfies $U = \coprod_{i=1}^r U_i$, then we have*

$$\int_U f(x) d\mu_c(x) = \sum_{i=1}^r \int_{U_i} f(x) d\mu_c(x).$$

Proof. By Lemma 4.15, we can find a decomposition

$$U_i = \coprod_{j=1}^{r_i} D(a_{i,j}, q^{-\nu_{i,j}})$$

with some $a_{i,j} \in \mathbb{P}^1(K_\infty)$ and $\nu_{i,j} \in \mathbb{Z}$. Then we have

$$U = \coprod_{i=1}^{r} \coprod_{j=1}^{r_i} D(a_{i,j}, q^{-\nu_{i,j}}).$$

Then Lemma 9.41 implies

$$\int_U f(x) d\mu_c(x) = \sum_{i=1}^{r} \sum_{j=1}^{r_i} \int_{D(a_{i,j}, q^{-\nu_{i,j}})} f(x) d\mu_c(x) = \sum_{i=1}^{r} \int_{U_i} f(x) d\mu_c(x).$$

□

Lemma 9.43. *For any integer $0 \leqslant i \leqslant k-2$ and $e \in \mathcal{T}_1^o$, we have*

$$\int_{U(e)} x^i d\mu_c(x) = (-1)^i c(e)(X^{k-2-i} Y^i).$$

Proof. By Lemma 4.15, we can write

$$U(e) = \coprod_{i=1}^{r} D(a_i, q^{-\nu_i})$$

with some $a_i \in \mathbb{P}^1(K_\infty)$ and $\nu_i \in \mathbb{Z}$. By Lemma 9.6 and Lemma 9.42, we may assume $U(e) = D(a, q^{-\nu})$ with some $a \in \mathbb{P}^1(K_\infty)$ and $\nu \in \mathbb{Z}$.

For any $\nu' \geqslant 1 + |\nu|$, we choose a decomposition

$$U(e) = \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}), \quad D(a', q^{-\nu'}) = U(e_{a'}).$$

Then we have $T_{a',\nu'}(x^i) = x^i$ and Lemma 9.6 yields

$$\begin{aligned} m_{a,\nu,\nu'}(x^i) &= \sum_{a' \in \Lambda_{\nu'}} \int_{U(e_{a'})} T_{a',\nu'}(x^i) d\mu_c(x) \\ &= \sum_{a' \in \Lambda_{\nu'}} (-1)^i c(e_{a'})(X^{k-2-i} Y^i) \\ &= (-1)^i c(e)(X^{k-2-i} Y^i). \end{aligned}$$

Taking the limit we obtain the lemma. □

Lemma 9.44. *Let $a \in K_\infty$ and $\nu \in \mathbb{Z}$. Let*

$$f(x) = \sum_{i \geqslant k-1} c_i (x-a)^i \in \mathcal{O}(D_{\mathbb{C}_\infty}(a, q^{-\nu})).$$

Let $C > 0$ be the constant of Lemma 9.13. Then for any $\nu' \geqslant \nu$, we have

$$|m_{a,\nu,\nu'}(f)| \leqslant C|\pi_\infty|^{(k-1)\nu - \frac{(k-2)\nu'}{2}}|I(f,a,\nu)|.$$

Proof. Choose a decomposition

$$D(a, q^{-\nu}) = \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}).$$

By the equality

$$(x-a)^i = \sum_{j=0}^{i} \binom{i}{j} (a'-a)^{i-j}(x-a')^j, \tag{9.9}$$

we have

$$T_{a',\nu'}(f) = \sum_{j=0}^{k-2} \sum_{i \geqslant k-1} \binom{i}{j} c_i (a'-a)^{i-j}(x-a')^j.$$

Then Lemma 9.13 yields

$$\begin{aligned}
|m_{a,\nu,\nu'}(f)| &= \left| \sum_{a' \in \Lambda_{\nu'}} \int_{D(a',q^{-\nu'})} T_{a',\nu'}(f) d\mu_c(x) \right| \\
&\leqslant \max_{a' \in \Lambda_{\nu'}} \max_{j=0,\ldots,k-2} \left| \sum_{i \geqslant k-1} c_i (a'-a)^{i-j} \right| C|\pi_\infty|^{(j-\frac{k-2}{2})\nu'} \\
&\leqslant \max_{j=0,\ldots,k-2} \left| \sum_{i \geqslant k-1} c_i \pi_\infty^{(i-j)\nu} \mathcal{O}_{\mathbb{C}_\infty} \right| C|\pi_\infty|^{(j-\frac{k-2}{2})\nu'} \\
&\leqslant \max_{j=0,\ldots,k-2} C|\pi_\infty|^{j(\nu'-\nu)+(k-1)\nu - \frac{(k-2)\nu'}{2}} |I(f,a,\nu)| \\
&= C|\pi_\infty|^{(k-1)\nu - \frac{(k-2)\nu'}{2}} |I(f,a,\nu)|.
\end{aligned}$$

$\square$

Lemma 9.45. *There exists a constant $C_1 > 0$ such that for any $e \in \mathcal{T}_1^o$ with $\infty \notin U(e)$, any $a \in U(e)$ and any integer $i \geqslant 0$, we have*

$$\left| \int_{U(e)} (x-a)^i d\mu_c(x) \right| \leqslant C_1 \rho(e)^{i - \frac{k-2}{2}}.$$

Proof. Write $U(e) = D(a, q^{-\nu})$ with some $\nu \in \mathbb{Z}$. Put $f = (x-a)^i$ and

$$m(f) = \int_{U(e)} (x-a)^i d\mu_c(x).$$

Lemma 9.13 shows that the constant C of this lemma satisfies the inequality of the lemma for $i \leqslant k-2$. Thus we may assume $i \geqslant k-1$.

To compute $m(f)$, for any $\nu' \geqslant \nu$ we choose a decomposition

$$U(e) = \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}).$$

Note that we have

$$I(f, a, \nu) = \pi_\infty^{(i-(k-1))\nu} \mathcal{O}_{\mathbb{C}_\infty}.$$

Since $i \geqslant k-1$, Lemma 9.44 yields

$$|m_{a,\nu,\nu'}(f)| \leqslant C|\pi_\infty|^{(k-1)\nu - \frac{(k-2)\nu'}{2}} |I(f, a, \nu)| = C|\pi_\infty|^{i\nu - \frac{(k-2)\nu'}{2}}.$$

On the other hand, by Lemma 9.28, for any $\nu'' > \nu' \geqslant \nu$ we have

$$\begin{aligned}
|m_{a,\nu,\nu''}(f) - m_{a,\nu,\nu'}(f)| &\leqslant \max_{l=0,\ldots,\nu''-\nu'-1} |m_{a,\nu,\nu'+l+1}(f) - m_{a,\nu,\nu'+l}(f)| \\
&\leqslant \max_{l=0,\ldots,\nu''-\nu'-1} C|\pi_\infty|^{-\frac{k-2}{2} + \frac{k(\nu'+l)}{2}} |I(f, a, \nu)| \\
&= C|\pi_\infty|^{-\frac{k-2}{2} + \frac{k\nu'}{2}} |I(f, a, \nu)| \\
&= C|\pi_\infty|^{-\frac{k-2}{2} + \frac{k\nu'}{2} + (i-(k-1))\nu}.
\end{aligned}$$

Taking $\nu'' \to \infty$, we obtain

$$|m(f) - m_{a,\nu,\nu'}(f)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2} + \frac{k\nu'}{2} + (i-(k-1))\nu}$$

for any $\nu' \geqslant \nu$.

For any $\nu' > \nu$, we have $(k-1)\nu < (k-1)\nu' - \frac{k-2}{2}$ and thus

$$i\nu - \frac{(k-2)\nu'}{2} < -\frac{k-2}{2} + \frac{k\nu'}{2} + (i-(k-1))\nu.$$

Putting $\nu' = \nu + 1$, this yields

$$|m(f)| \leqslant C|\pi_\infty|^{(i - \frac{k-2}{2})\nu - \frac{k-2}{2}}.$$

Since $\rho(e) = |\pi_\infty|^\nu$, the constant $C|\pi_\infty|^{-\frac{k-2}{2}}$ satisfies the condition for any $i \geqslant k-1$. Thus we may put

$$C_1 = \max\{C, C|\pi_\infty|^{-\frac{k-2}{2}}\} = C|\pi_\infty|^{-\frac{k-2}{2}}.$$

□

Lemma 9.46. *Let $\nu \in \mathbb{Z}$ and let*

$$f(x) = \sum_{i \geqslant 1} \frac{c_i}{x^i} \in \mathcal{O}(D_{\mathbb{C}_\infty}(\infty, q^{-\nu})).$$

Let $C > 0$ be the constant of Lemma 9.13. Then for any $\nu' \geqslant 1 + |\nu|$, we have

$$|m_{\infty,\nu,\nu'}(f)| \leqslant C|\pi_\infty|^{\frac{k\nu}{2} - \frac{k-2}{2}} |I(f, \infty, \nu)|.$$

Proof. Choose a decomposition

$$D(\infty, q^{-\nu}) = \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}).$$

Since the assumption on f implies $T_{\infty,\nu'}(f) = 0$, we have

$$m_{\infty,\nu,\nu'}(f) = \sum_{a' \in \Lambda_{\nu'} \backslash \{\infty\}} \int_{D(a', q^{-\nu'})} T_{a',\nu'}(f) d\mu_c(x).$$

By Lemma 4.14, we can find a subset $\Lambda \subseteq \Lambda_{\nu'} \backslash \{\infty\}$ satisfying

$$D(\infty, q^{-\nu}) \backslash D(\infty, q^{-\nu'}) = \coprod_{a' \in \Lambda} D(a', q^{\nu-1}) = \coprod_{a' \in \Lambda_{\nu'} \backslash \{\infty\}} D(a', q^{-\nu'}),$$

where the latter covering is a refinement of the former. By using Lemma 9.35 repeatedly, we have

$$m_{\infty,\nu,\nu'}(f) \in \sum_{a' \in \Lambda} \int_{D(a', q^{\nu-1})} T_{a',1-\nu}(f) d\mu_c(x) + I$$

with some monogenic $\mathcal{O}_{\mathbb{C}_\infty}$-submodule I of $\mathbb{C}_\infty$ satisfying

$$|I| \leqslant C|\pi_\infty|^{1+\frac{k}{2}\nu} |I(f, \infty, \nu)|.$$

By the equality

$$x^{-i} = \sum_{j \geqslant 0} \binom{-i}{j} \frac{1}{(a')^{i+j}} (x - a')^j, \tag{9.10}$$

we have

$$T_{a',1-\nu}(f) = \sum_{j=0}^{k-2} \sum_{i \geqslant 1} \binom{-i}{j} \frac{c_i}{(a')^{i+j}} (x - a')^j.$$

Since $|a'| \geqslant q^\nu$, Lemma 9.13 yields

$$\begin{aligned}
&\left| \sum_{a' \in \Lambda} \int_{D(a', q^{\nu-1})} T_{a',1-\nu}(f) d\mu_c(x) \right| \\
&\quad \leqslant \max_{a' \in \Lambda} \max_{j=0,\dots,k-2} \left| \sum_{i \geqslant 1} c_i \pi_\infty^{(i+j)\nu} \mathcal{O}_{\mathbb{C}_\infty} \right| C|\pi_\infty|^{(1-\nu)(j - \frac{k-2}{2})} \\
&\quad \leqslant \max_{j=0,\dots,k-2} C|\pi_\infty|^{\nu + j - \frac{k-2}{2} + \frac{(k-2)\nu}{2}} |I(f, \infty, \nu)| \\
&\quad = C|\pi_\infty|^{\frac{k}{2}\nu - \frac{k-2}{2}} |I(f, \infty, \nu)|.
\end{aligned}$$

Since $k \geqslant 2$, we obtain

$$\begin{aligned}
|m_{\infty,\nu,\nu'}(f)| &\leqslant C \max\{|\pi_\infty|^{1+\frac{k}{2}\nu}, |\pi_\infty|^{\frac{k}{2}\nu - \frac{k-2}{2}}\} |I(f, \infty, \nu)| \\
&= C|\pi_\infty|^{\frac{k}{2}\nu - \frac{k-2}{2}} |I(f, \infty, \nu)|.
\end{aligned}$$

This concludes the proof of the lemma. $\square$

Lemma 9.47. *There exists a constant $C_2 > 0$ such that for any $e \in \mathcal{T}_1^o$ with $\infty \in U(e)$ and $0 \notin U(e)$ and any integer $i \geqslant -(k-2)$, we have*

$$\left|\int_{U(e)} \frac{1}{x^i} d\mu_c(x)\right| \leqslant C_2 \rho(e)^{i+\frac{k-2}{2}}.$$

Proof. By Lemma 9.15, we can write $U(e) = D(\infty, q^{-\nu})$ with some $\nu \in \mathbb{Z}$. Then we have $\rho(e) = q^{-\nu}$ and

$$U(-e) = D(0, q^{\nu-1}), \quad \rho(-e) = q^{\nu-1} = q^{-1}\rho(e)^{-1}.$$

Put $f = \frac{1}{x^i}$ and

$$m(f) = \int_{U(e)} \frac{1}{x^i} d\mu_c(x).$$

For $0 \geqslant i \geqslant -(k-2)$, Lemma 9.13 and (9.2) show that the constant C of this lemma satisfies

$$\begin{aligned}\left|\int_{U(e)} x^{-i} d\mu_c(x)\right| &= \left|\int_{U(-e)} x^{-i} d\mu_c(x)\right| \\ &< C\rho(-e)^{-i-\frac{k-2}{2}} = Cq^{i+\frac{k-2}{2}}\rho(e)^{i+\frac{k-2}{2}} \leqslant Cq^{\frac{k-2}{2}}\rho(e)^{i+\frac{k-2}{2}}\end{aligned}$$

and the constant $C_2 = C|\pi_\infty|^{-\frac{k-2}{2}}$ satisfies the inequality of the lemma for $0 \geqslant i \geqslant -(k-2)$.

Suppose $i \geqslant 1$. To compute $m(f)$, for any $\nu' \geqslant 1 + |\nu|$ we choose a decomposition

$$U(e) = \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}).$$

Note that we have

$$I(f, \infty, \nu) = \pi_\infty^{(i-1)\nu} \mathcal{O}_{\mathbb{C}_\infty}.$$

Then Lemma 9.46 yields

$$|m_{\infty,\nu,\nu'}(f)| \leqslant C|\pi_\infty|^{\frac{k}{2}\nu - \frac{k-2}{2}} |I(f, \infty, \nu)| = C|\pi_\infty|^{(\frac{k}{2}+i-1)\nu - \frac{k-2}{2}}.$$

Taking $\nu' \to \infty$, we obtain

$$\begin{aligned}|m(f)| &\leqslant C|\pi_\infty|^{(\frac{k}{2}+i-1)\nu - \frac{k-2}{2}} = C|\pi_\infty|^{(i+\frac{k-2}{2})\nu - \frac{k-2}{2}} \\ &= C|\pi_\infty|^{-\frac{k-2}{2}}\rho(e)^{i+\frac{k-2}{2}}.\end{aligned}$$

Thus the constant $C_2 = C|\pi_\infty|^{-\frac{k-2}{2}}$ satisfies the inequality of the lemma also for $i \geqslant 1$. $\square$

Thus the constant $\max\{C_1, C_2\}$ satisfies the condition (3) of Theorem 9.16.

Lemma 9.48. *Let $e \in \mathcal{T}_1^o$ satisfying $\infty \notin U(e)$. Write $U(e) = D(a, q^{-\nu})$ with some $a \in K_\infty$ and $\nu \in \mathbb{Z}$. Let $F(x) = \sum_{i \geqslant 0} c_i(x-a)^i \in \mathbb{C}_\infty\langle \frac{x-a}{\pi_\infty^\nu} \rangle$. Then we have*

$$\int_{U(e)} F(x) d\mu_c(x) = \sum_{i \geqslant 0} c_i \int_{U(e)} (x-a)^i d\mu_c(x).$$

Proof. For any $l \geqslant k-1$, put

$$F_l(x) = \sum_{i \geqslant l} c_i(x-a)^i, \quad m(F_l) = \int_{U(e)} F_l(x) d\mu_c(x).$$

It is enough to show $\lim_{l\to\infty} m(F_l) = 0$.

To compute $m(F_l)$, for any $\nu' \geqslant \nu$ we choose a decomposition

$$U(e) = \coprod_{a' \in \Lambda_{\nu'}} D(a', q^{-\nu'}).$$

By Lemma 9.44, we have

$$|m_{a,\nu,\nu'}(F_l)| \leqslant C|\pi_\infty|^{(k-1)\nu - \frac{(k-2)\nu'}{2}} |I(F_l, a, \nu)|.$$

On the other hand, as in the proof of Lemma 9.45, Lemma 9.28 shows

$$|m(F_l) - m_{a,\nu,\nu'}(F_l)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2} + \frac{k\nu'}{2}} |I(F_l, a, \nu)|.$$

When $\nu' \geqslant \nu + 1$, we have $\nu + \frac{k-2}{2(k-1)} \leqslant \nu'$ and

$$(k-1)\nu - \frac{(k-2)\nu'}{2} \leqslant -\frac{k-2}{2} + \frac{k\nu'}{2}.$$

For $\nu' = \nu + 1$, this implies

$$|m(F_l)| \leqslant C|\pi_\infty|^{(k-1)\nu - \frac{(k-2)\nu'}{2}} |I(F_l, a, \nu)| = C|\pi_\infty|^{\frac{k\nu}{2} - \frac{k-2}{2}} |I(F_l, a, \nu)|.$$

Since $\lim_{i\to\infty} c_i \pi_\infty^{i\nu} = 0$, we have $\lim_{l\to\infty} |I(F_l, a, \nu)| = 0$. This concludes the proof. $\square$

Lemma 9.49. *Let $e \in \mathcal{T}_1^o$ satisfying $\infty \in U(e)$ and $0 \notin U(e)$. Write $U(e) = D(\infty, q^{-\nu})$ with some $\nu \in \mathbb{Z}$. Let $F(x) = \sum_{i \geqslant -(k-2)} \frac{c_i}{x^i} \in x^{k-2}\mathbb{C}_\infty\langle \frac{1}{\pi_\infty^\nu x} \rangle$. Then we have*

$$\int_{U(e)} F(x) d\mu_c(x) = \sum_{i \geqslant -(k-2)} c_i \int_{U(e)} \frac{1}{x^i} d\mu_c(x).$$

Proof. For any $l \geqslant 1$, put

$$F_l(x) = \sum_{i \geqslant l} \frac{c_i}{x^i}, \quad m(F_l) = \int_{U(e)} F_l(x) d\mu_c(x).$$

It is enough to show $\lim_{l\to\infty} m(F_l) = 0$.

By Lemma 9.46 and taking $\nu' \to \infty$, we have

$$|m(F_l)| \leqslant C|\pi_\infty|^{\frac{k}{2}\nu - \frac{k-2}{2}}|I(F_l, \infty, \nu)|.$$

Since $\lim_{i\to\infty} c_i \pi_\infty^{i\nu} = 0$, we have $\lim_{l\to\infty} |I(F_l, \infty, \nu)| = 0$. This concludes the proof. □

Now the proof of Theorem 9.16 is completed.

9.7. **Transformation property of the integration.** For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$ and any rigid analytic function f on a distinguished closed disc in $\mathbb{P}^1(\mathbb{C}_\infty)$, we define

$$f_\gamma(x) := \det(\gamma)^{2-k}(cx+d)^{k-2} f\left(\frac{ax+b}{cx+d}\right).$$

Note that if $f \in P_k$, then $f_\gamma \in P_k$.

Since $\gamma : \mathbb{P}^1(\mathbb{C}_\infty) \to \mathbb{P}^1(\mathbb{C}_\infty)$ is an isomorphism of rigid analytic varieties, for any $e \in \mathcal{T}_1^o$ its restriction to the open subvariety $\mathcal{U}(e)$ induces an isomorphism of affinoid algebras over $\mathbb{C}_\infty$

$$\gamma^* : \mathcal{O}(\mathcal{U}(\gamma \circ e)) \to \mathcal{O}(\mathcal{U}(e)), \quad f \mapsto \gamma^*(f)(x) = f\left(\frac{ax+b}{cx+d}\right). \tag{9.11}$$

Note that for any $e \in \mathcal{T}_1^o$, the ring $\mathcal{O}(\mathcal{U}(e))$ is a PID. Hence for any $z \in \mathcal{U}(e)$ and $g \in \mathrm{Frac}(\mathcal{O}(\mathcal{U}(e)))$, we may define the vanishing order $\mathrm{ord}_z(g)$ of g at z. Then, for any $z \in \mathcal{U}(e)$ and $f \in \mathcal{O}(\mathcal{U}(\gamma \circ e))$, we have

$$\mathrm{ord}_z(\gamma^*(f)) = \mathrm{ord}_{\gamma(z)}(f). \tag{9.12}$$

For any $e \in \mathcal{T}_1^o$ satisfying $\infty \notin U(e)$ or $0 \notin U(e)$, write $\rho(e) = q^{-\nu(e)}$ and put

$$m(e) = \begin{cases} 0 & (\infty \notin U(e)), \\ k-2 & (\infty \in U(e) \text{ and } 0 \notin U(e)), \end{cases}$$

so that $P_k \subseteq x^{m(e)}\mathcal{O}(\mathcal{U}(e))$. When $\infty \notin U(e)$, choose any element $z(e) \in U(e)$. When $\infty \in U(e)$, we put $z(e) = \infty$.

Lemma 9.50. *Let $e \in \mathcal{T}_1^o$ and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$. Let $f \in x^{m(\gamma\circ e)}\mathcal{O}(\mathcal{U}(\gamma \circ e))$.*

(1) Suppose $\infty \notin D$ or $0 \notin D$ for any $D \in \{U(e), U(\gamma \circ e)\}$. Then

$$f_\gamma \in x^{m(e)}\mathcal{O}(\mathcal{U}(e)).$$

(2) Suppose moreover $\gamma(z(e)) = z(\gamma \circ e)$. Then

$$T_{z(e),\nu(e)}(f_\gamma) = T_{z(\gamma\circ e),\nu(\gamma\circ e)}(f)_\gamma.$$

Proof. First we claim that

$$J(e,\gamma) := x^{-m(e)}\left(\frac{ax+b}{cx+d}\right)^{m(\gamma\circ e)}(cx+d)^{k-2} \in \mathcal{O}(\mathcal{U}(e))^{\times}.$$

Indeed, if $\infty \notin U(e)$ and $\infty \notin U(\gamma\circ e)$, then $J(e,\gamma) = (cx+d)^{k-2} \in \mathcal{O}(\mathcal{U}(e))^{\times}$ since $\frac{-d}{c} = \gamma^{-1}(\infty) \notin \mathcal{U}(e)$. If $\infty \notin U(e)$ and $\infty \in U(\gamma\circ e)$, then $J(e,\gamma) = (ax+b)^{k-2} \in \mathcal{O}(\mathcal{U}(e))^{\times}$ since $\frac{-b}{a} = \gamma^{-1}(0) \notin \mathcal{U}(e)$. If $\infty \in U(e)$ and $\infty \notin U(\gamma\circ e)$, then $J(e,\gamma) = (c+\frac{d}{x})^{k-2} \in \mathcal{O}(\mathcal{U}(e))^{\times}$ since $\frac{-d}{c} = \gamma^{-1}(\infty) \notin \mathcal{U}(e)$. If $\infty \in U(e)$ and $\infty \in U(\gamma\circ e)$, then $J(e,\gamma) = (a+\frac{b}{x})^{k-2} \in \mathcal{O}(\mathcal{U}(e))^{\times}$ since $\frac{-b}{a} = \gamma^{-1}(0) \notin \mathcal{U}(e)$.

By this claim, the map γ^* of (9.11) induces a $\mathbb{C}_\infty$-linear isomorphism

$$\begin{aligned} x^{m(\gamma\circ e)}\mathcal{O}(\mathcal{U}(\gamma\circ e)) &\to x^{m(e)}\mathcal{O}(\mathcal{U}(e)), \\ f = x^{m(\gamma\circ e)}g &\mapsto f_\gamma = x^{m(e)}\det(\gamma)^{2-k}J(e,\gamma)\gamma^* g \end{aligned}$$

and the assertion (1) follows.

Moreover, if the condition of (2) is satisfied, then the claim and (9.12) show

$$\mathrm{ord}_{z(\gamma\circ e)}(f) \geqslant k-1-m(\gamma\circ e) \quad \Leftrightarrow \quad \mathrm{ord}_{z(e)}(f_\gamma) \geqslant k-1-m(e).$$

Hence, if we write

$$f = T_{z(\gamma\circ e),\nu(\gamma\circ e)}(f) + F, \quad \mathrm{ord}_{z(\gamma\circ e)}(F) \geqslant k-1-m(\gamma\circ e),$$

then we have

$$f_\gamma = T_{z(\gamma\circ e),\nu(\gamma\circ e)}(f)_\gamma + F_\gamma, \quad \mathrm{ord}_{z(e)}(F_\gamma) \geqslant k-1-m(e).$$

Since $T_{z(\gamma\circ e),\nu(\gamma\circ e)}(f)_\gamma \in P_k$, we obtain $T_{z(\gamma\circ e),\nu(\gamma\circ e)}(f)_\gamma = T_{z(e),\nu(e)}(f_\gamma)$. □

Lemma 9.51. *For any $f \in \mathscr{A}_k$ and any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$, we have*

$$f_\gamma(x) := \det(\gamma)^{2-k}(cx+d)^{k-2}f\left(\frac{ax+b}{cx+d}\right) \in \mathscr{A}_k.$$

Proof. We show that f_γ satisfies the condition of Definition 9.14 for all $z \in \mathbb{P}^1(K_\infty)$. By Lemma 9.50 (1), it is enough to show that for any $z \in \mathbb{P}^1(K_\infty)$, there exists an integer ν such that $D = D(z, q^{-\nu})$ satisfies $\infty \notin \gamma(D)$ or $0 \notin \gamma(D)$.

Suppose $0, \infty \in \gamma(D(z, q^{-\nu}))$ for some $\nu \in \mathbb{Z}$. Since $\gamma^{-1}(0) \neq \gamma^{-1}(\infty)$, we can find $\nu' \in \mathbb{Z}$ with $\nu' \geqslant \nu$ such that the set $\{\gamma^{-1}(0), \gamma^{-1}(\infty)\} \cap D(z, q^{-\nu'})$ consists of at most one element. This ν' satisfies the requirement. □

Lemma 9.52. *Let $f \in \mathscr{A}_k$ and let $\gamma_1, \gamma_2 \in GL_2(K)$. Then we have*

$$f_{\gamma_1\gamma_2} = (f_{\gamma_1})_{\gamma_2}, \quad f_{\mathrm{id}} = f.$$

Proof. Write $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Since $\gamma_1(\gamma_2(x)) = (\gamma_1\gamma_2)(x)$, we have

$$\begin{aligned}
(f_{\gamma_1})_{\gamma_2}(x) &= \det(\gamma_2)^{2-k}(c_2x+d_2)^{k-2}f_{\gamma_1}(\gamma_2(x)) \\
&= \det(\gamma_1)^{2-k}\det(\gamma_2)^{2-k}\cdot \\
&\qquad (c_2x+d_2)^{k-2}\left(c_1\left(\frac{a_2x+b_2}{c_2x+d_2}\right)+d_1\right)^{k-2} f(\gamma_1(\gamma_2(x))) \\
&= \det(\gamma_1\gamma_2)^{2-k}((c_1a_2+d_1c_2)x+(c_1b_2+d_1d_2))^{k-2}f((\gamma_1\gamma_2)(x)) \\
&= f_{\gamma_1\gamma_2}(x).
\end{aligned}$$

Thus we obtain the first equality of the lemma. The second equality is clear. □

Lemma 9.53. *Let $f \in \mathscr{A}_k$, $e \in \mathcal{T}_1^o$ and $\gamma \in GL_2(K)$. Suppose $\infty \notin U(e)$ and $\infty \notin U(\gamma \circ e)$. Then we have*

$$\int_{U(\gamma\circ e)} f(x)d\mu_{\gamma c}(x) = \int_{U(e)} f_\gamma(x)d\mu_c(x).$$

Proof. By Lemma 9.8, the function $z \mapsto |cz+d|$ is constant on $U(e)$. Since $\infty \notin U(\gamma \circ e)$, we have $cz+d \neq 0$ for any $z \in U(e)$ and this constant is positive. Let q^m be this constant and write $|\det(\gamma)| = q^{m_0}$.

For any sufficiently large integer ν' satisfying $q^m > q^{-\nu'}|c|$, take any decomposition

$$U(e) = \coprod_{z\in\Lambda_{\nu'}} D(z, q^{-\nu'})$$

as in (9.4). Then Lemma 4.8 and Lemma 9.50 (2) imply $\gamma(D(z,q^{-\nu'})) = D(\gamma(z), q^{-(\nu'+2m-m_0)})$ and

$$U(\gamma\circ e) = \coprod_{z\in\Lambda_{\nu'}} D(\gamma(z), q^{-(\nu'+2m-m_0)}), \quad T_{z,\nu'}(f_\gamma) = T_{\gamma(z),\nu'+2m-m_0}(f)_\gamma.$$

By Lemma 9.7 (1), we have

$$\begin{aligned}
m_{\nu'+2m-m_0}(f) &:= \sum_{z\in\Lambda_{\nu'}} \int_{D(\gamma(z),q^{-(\nu'+2m-m_0)})} T_{\gamma(z),\nu'+2m-m_0}(f)d\mu_{\gamma c}(x) \\
&= \sum_{z\in\Lambda_{\nu'}} \int_{D(z,q^{-\nu'})} T_{\gamma(z),\nu'+2m-m_0}(f)_\gamma d\mu_c(x) \\
&= \sum_{z\in\Lambda_{\nu'}} \int_{D(z,q^{-\nu'})} T_{z,\nu'}(f_\gamma)d\mu_c(x) =: m_{\nu'}(f_\gamma).
\end{aligned}$$

Now the lemma follows from

$$\int_{U(\gamma\circ e)} f(x)d\mu_{\gamma c}(x) = \lim_{\nu'\to\infty} m_{\nu'+2m-m_0}(f), \quad \int_{U(e)} f_\gamma(x)d\mu_c(x) = \lim_{\nu'\to\infty} m_{\nu'}(f_\gamma).$$

□

Lemma 9.54. *Let $f \in \mathscr{A}_k$, $e \in \mathcal{T}_1^o$ and $\gamma \in GL_2(K)$. Suppose $\infty \in U(e)$ and $\infty \notin U(\gamma \circ e)$. Then we have*

$$\int_{U(\gamma\circ e)} f(x)d\mu_{\gamma c}(x) = \int_{U(e)} f_\gamma(x)d\mu_c(x).$$

Proof. Since $\infty \in U(e)$ and $\infty \notin U(\gamma\circ e)$, we have $c \neq 0$ and $\frac{a}{c} \in U(\gamma\circ e)$. Take a sufficiently large positive integer ν satisfying $|d| < q^\nu|c|$ and a decomposition

$$U(e) = D\left(\infty, q^{-\nu}\right) \sqcup \coprod_{z\in\Lambda} D(z, q^{-\nu}).$$

By Lemma 9.53, we have

$$\int_{\gamma(D(z,q^{-\nu}))} f(x)d\mu_{\gamma c}(x) = \int_{D(z,q^{-\nu})} f_\gamma(x)d\mu_c(x)$$

for any $z \in \Lambda$. Thus we may assume $U(e) = D\left(\infty, q^{-\nu}\right)$.

For any sufficiently large positive integer $\nu' \geqslant \nu$, take any decomposition

$$U(e) = D(\infty, q^{-\nu'}) \sqcup \coprod_{z\in\Lambda_{\nu'}} D(z, q^{-\nu'})$$

as in Definition 9.34.

Write $|c| = q^t$, $|\det(\gamma)| = q^{m_0}$ and $|z| = q^{s_z}$ for any $z \in \Lambda_{\nu'}$. Put $t_0 = 2t - m_0$. Then we have

$$0 < \nu \leqslant s_z \leqslant \nu' - 1 \tag{9.13}$$

and $|d| < q^\nu|c| \leqslant |cz|$, which yields $|cz + d| = |cz|$ and

$$|cz + d| = |cz| \geqslant q^\nu|c| > q^{-\nu}|c|.$$

By Lemma 4.8, Lemma 4.9 and Lemma 9.50 (2), we have $\gamma(D(\infty, q^{-\nu})) = D(\frac{a}{c}, q^{-\nu-t_0})$ and

$$\gamma(D(\infty, q^{-\nu'})) = D\left(\frac{a}{c}, q^{-\nu'-t_0}\right), \quad T_{\infty,\nu'}(f_\gamma) = T_{\frac{a}{c},\nu'+t_0}(f)_\gamma,$$
$$\gamma(D(z, q^{-\nu'})) = D(\gamma(z), q^{-\nu'-t_0-2s_z}), \quad T_{z,\nu'}(f_\gamma) = T_{\gamma(z),\nu'+t_0+2s_z}(f)_\gamma$$

for any $z \in \Lambda_{\nu'}$.

Put

$$m_{\nu'}(f_\gamma) := \int_{D(\infty,q^{-\nu'})} T_{\infty,\nu'}(f_\gamma)d\mu_c(x) + \sum_{z\in\Lambda_{\nu'}} \int_{D(z,q^{-\nu'})} T_{z,\nu'}(f_\gamma)d\mu_c(x),$$

$$\begin{aligned}\tilde{m}_{\nu'}(f) := &\int_{D\left(\frac{a}{c},q^{-\nu'-t_0}\right)} T_{\frac{a}{c},\nu'+t_0}(f)d\mu_{\gamma_c}(x)\\ &+ \sum_{z\in\Lambda_{\nu'}} \int_{D(\gamma(z),q^{-\nu'-t_0-2s_z})} T_{\gamma(z),\nu'+t_0+2s_z}(f)d\mu_{\gamma_c}(x).\end{aligned}$$

By Lemma 9.7 (1), we have $m_{\nu'}(f_\gamma) = \tilde{m}_{\nu'}(f)$ and

$$\int_{U(e)} f_\gamma(x)d\mu_c(x) = \lim_{\nu'\to\infty} m_{\nu'}(f_\gamma).$$

For $\tilde{m}_{\nu'}(f)$, (9.13) implies

$$\nu' + t_0 \leqslant \nu' + t_0 + 2s_z \leqslant 3\nu' + t_0 - 2.$$

Take any decomposition

$$\begin{aligned} D\left(\frac{a}{c}, q^{-\nu'-t_0}\right) &= \coprod_{w\in\Lambda(\infty)} D(w, q^{-(3\nu'+t_0-2)}),\\ D(\gamma(z), q^{-\nu'-t_0-2s_z}) &= \coprod_{w\in\Lambda(z)} D(w, q^{-(3\nu'+t_0-2)})\end{aligned}$$

for any $z \in \Lambda_{\nu'}$. Put $\Lambda_{3\nu'+t_0-2} = \Lambda(\infty) \cup \bigcup_{z\in\Lambda_{\nu'}} \Lambda(z)$ and

$$m_{3\nu'+t_0-2}(f) := \sum_{w\in\Lambda_{3\nu'+t_0-2}} \int_{D(w,q^{-(3\nu'+t_0-2)})} T_{w,3\nu'+t_0-2}(f)d\mu_{\gamma_c}(x).$$

Since this agrees with the sum of Definition 9.25, we have

$$\int_{U(\gamma\circ e)} f(x)d\mu_{\gamma_c}(x) = \lim_{\nu'\to\infty} m_{3\nu'+t_0-2}(f).$$

Now Lemma 9.27 yields

$$|\tilde{m}_{\nu'}(f) - m_{3\nu'+t_0-2}(f)| \leqslant C|\pi_\infty|^{-\frac{k-2}{2}+\frac{k}{2}(\nu'+t_0)} \left|I\left(f, \frac{a}{c}, \nu + t_0\right)\right|.$$

with some constant $C > 0$. Since $k > 0$, we have

$$\lim_{\nu'\to\infty} \tilde{m}_{\nu'}(f) = \lim_{\nu'\to\infty} m_{3\nu'+t_0-2}(f) = \int_{U(\gamma\circ e)} f(x)d\mu_{\gamma_c}(x).$$

This concludes the proof of the lemma. $\square$

Proposition 9.55. *Let $f \in \mathscr{A}_k$ and let $e \in \mathcal{T}_1^o$. Then for any $\gamma \in GL_2(K)$, we have*
$$\int_{U(\gamma\circ e)} f(x)d\mu_{{}^{\gamma}c}(x) = \int_{U(e)} \det(\gamma)^{2-k}(cx+d)^{k-2} f\left(\frac{ax+b}{cx+d}\right) d\mu_c(x).$$

Proof. By Lemma 9.53 and Lemma 9.54, we may assume $\infty \in U(\gamma \circ e)$. For a sufficiently large positive integer ν, we have a decomposition
$$U(\gamma\circ e) = D(\infty, q^{-\nu}) \sqcup \coprod_{z\in\Lambda} D(z, q^{-\nu}).$$
Since $\infty \notin D(z, q^{-\nu})$, Lemma 9.53 and Lemma 9.54 imply
$$\int_{D(z,q^{-\nu})} f(x)d\mu_{c^{\gamma}}(x) = \int_{\gamma^{-1}(D(z,q^{-\nu}))} f_{\gamma}(x)d\mu_c(x).$$
By Theorem 9.16 (1), we may assume $U(\gamma\circ e) = D(\infty, q^{-\nu})$.

Take any $Q \neq 0 \in A$ satisfying $\deg(Q) > -\nu$ and put
$$\delta := \begin{pmatrix} 1 & 0 \\ Q & 1 \end{pmatrix} \in GL_2(K).$$
Then $Qz + 1 \neq 0$ for any $z \in K_\infty$ satisfying $|z| \geqslant q^{\nu}$, so that $\infty \notin \delta(U(\gamma\circ e)) = U(\delta\gamma\circ e)$. Lemma 9.51 yields $f_{\delta^{-1}} \in \mathscr{A}_k$.

Applying Lemma 9.53 and Lemma 9.54 to $f_{\delta^{-1}}$ and the map
$$\delta\gamma : U(e) \to U(\delta\gamma\circ e),$$
Lemma 9.52 gives
$$\int_{U(\delta\gamma\circ e)} f_{\delta^{-1}}(x)d\mu_{{}^{\delta\gamma}c}(x) = \int_{U(e)} (f_{\delta^{-1}})_{\delta\gamma}(x)d\mu_c(x) = \int_{U(e)} f_{\gamma}(x)d\mu_c(x).$$
Similarly, for the map
$$\delta : U(\gamma\circ e) \to U(\delta\gamma\circ e),$$
since ${}^{\delta\gamma}c = {}^{\delta}({}^{\gamma}c)$ we have
$$\int_{U(\delta\gamma\circ e)} f_{\delta^{-1}}(x)d\mu_{{}^{\delta\gamma}c}(x) = \int_{U(\gamma\circ e)} (f_{\delta^{-1}})_{\delta}(x)d\mu_{{}^{\gamma}c}(x) = \int_{U(\gamma\circ e)} f(x)d\mu_{{}^{\gamma}c}(x).$$
Hence the proposition follows. $\square$

10. Residue theorems

In this section, we recall the theory of rigid analytic residues on $\mathbb{P}^1$, following [FvdP1, §I.3]. Let $\mathbb{K}$ be an algebraically closed field equipped with a complete non-Archimedean valuation $|-| : \mathbb{K} \to \mathbb{R}_{\geqslant 0}$. Let $\mathcal{O}_{\mathbb{K}}$ be its ring of integers, $m_{\mathbb{K}}$ be the maximal ideal of $\mathcal{O}_{\mathbb{K}}$ and k be its residue field. Since k is also algebraically closed, we see that k is an infinite field.

10.1. **Circular residue.**

Lemma 10.1. *Let* $\mathrm{Sp}(R)$ *be a connected affinoid subdomain of* $\mathrm{Sp}(\mathbb{K}\langle x\rangle)$. *Then* R *is a PID of dimension one such that its prime element is* $x-c$ *with some* $c \in \mathbb{K}$ *satisfying* $|c| \leqslant 1$.

Proof. By [BGR, Theorem 5.2.6/1 and Remark 6.1.3], we see that $\mathbb{K}\langle x\rangle$ is a Noetherian UFD of dimension one. Hence it is a PID. Then [BGR, Proposition 7.2.2/1] shows that R is a regular ring of dimension one such that any maximal ideal of it is generated by $x-c$ with $|c| \leqslant 1$. Since $\mathrm{Sp}(R)$ is connected, it follows that R is a PID. □

Definition 10.2. Let $a \in \mathbb{K}$ and $\rho \in |\mathbb{K}^\times|$. Let $\varpi_\rho \in \mathbb{K}$ be any element satisfying $|\varpi_\rho| = \rho$. We define

$$C(a,\rho) = \{z \in \mathbb{K} \mid |z-a| = \rho\} = \mathrm{Sp}(\mathbb{K}\langle \frac{x-a}{\varpi_\rho}, \frac{\varpi_\rho}{x-a}\rangle)$$

and call it the circle centered at a with radius ρ. We also put

$$C_0 = C(0,1) = \{z \in \mathbb{K} \mid |z| = 1\} = \mathrm{Sp}(\mathbb{K}\langle x, x^{-1}\rangle).$$

By definition any element $f \in \mathcal{O}(C(a,\rho))$ is uniquely written as

$$f = \sum_{n\in\mathbb{Z}} a_n \left(\frac{x-a}{\varpi_\rho}\right)^n$$

with some $a_n \in \mathbb{K}$ satisfying $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{-n} = 0$. Then the ring $\mathcal{O}(C(a,\rho))$ is equipped with the Banach norm

$$|f| = \max\{|a_n| \mid n \in \mathbb{Z}\}$$

so that we have an isometric isomorphism of affinoid algebras over $\mathbb{K}$

$$\mathbb{K}\langle x, x^{-1}\rangle \to \mathbb{K}\langle \frac{x-a}{\varpi_\rho}, \frac{\varpi_\rho}{x-a}\rangle, \quad x \mapsto \frac{x-a}{\varpi_\rho}. \tag{10.1}$$

Lemma 10.3. *The Banach norm* $|-|$ *on* $\mathcal{O}(C(a,\rho))$ *agrees with the supremum norm and it is a valuation.*

Proof. Using the isometry (10.1), we reduce ourselves to showing the lemma for the unit circle C_0.

By a remark after [BGR, Proposition 6.1.4/2], the Banach norm is the same as the residue norm with respect to the surjection

$$\mathbb{K}\langle X, Y\rangle \to \mathbb{K}\langle x, x^{-1}\rangle, \quad X \mapsto x,\ Y \mapsto x^{-1}.$$

Since $k[x,x^{-1}]$ is a domain, [BGR, Proposition 6.4.3/4] implies that this norm agrees with the supremum norm. By [BGR, Proposition 6.2.3/5], it is also a valuation. □

Definition 10.4. Let $C = C(a, \rho)$ be a circle. We say $t \in \mathcal{O}(C)^\times$ is a parameter of C if the following conditions are satisfied:

(1) $|t|_{\sup} = 1$.
(2) Any element $f \in \mathcal{O}(C)$ can be written uniquely as
$$f = \sum_{n \in \mathbb{Z}} a_n t^n,$$
where $a_n \in \mathbb{K}$ satisfies $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{-n} = 0$.
(3) With a_n as above, we have $|f|_{\sup} = \max\{|a_n| \mid n \in \mathbb{Z}\}$.

Then t^{-1} is also a parameter of C.

By Lemma 10.3, the element $\frac{x-a}{\varpi_\rho} \in \mathcal{O}(C(a,\rho))$ is a parameter of $C(a, \rho)$.

Let C be a circle and let t be any parameter of C. Then the canonical reduction of the affinoid algebra $\mathcal{O}(C)$ is given by
$$\widetilde{\mathcal{O}(C)} = k[t, t^{-1}].$$
Thus we have an isomorphism of groups
$$t^{\mathbb{Z}} \to \widetilde{\mathcal{O}(C)}^\times / k^\times. \tag{10.2}$$

Definition 10.5. An orientation of a circle C is an isomorphism of groups $\psi : \mathbb{Z} \to \widetilde{\mathcal{O}(C)}^\times / k^\times$. We call the pair (C, ψ) of the circle C and an orientation ψ of C an oriented circle. For an orientation ψ of C, the isomorphism $\bar{\psi}(n) := \psi(-n)$ is called the orientation opposite to ψ. A parameter t of the oriented circle (C, ψ) is said to be positive if $\psi^{-1}(t)$ is positive.

Lemma 10.6. *Let t be a parameter of the circle C. Then $f \in \mathcal{O}(C)^\times$ if and only if*
$$f = \lambda t^n (1 + \sum_{i \neq 0} b_i t^i)$$
with $\lambda \in \mathbb{K}^\times$ and $b_i \in \mathbb{K}$ satisfying $|b_i| < 1$ for any i.

Proof. Suppose that f is written in the form as in the lemma. Since the series
$$(1 + \sum_{i \neq 0} b_i t^i)^{-1} = \sum_{j \geqslant 0} \left(- \sum_{i \neq 0} b_i t^i \right)^j$$
converges in $\mathcal{O}(C)$, we obtain $f \in \mathcal{O}(C)^\times$.

Conversely, take any $f \in \mathcal{O}(C)^\times$. By Lemma 10.3, the Banach norm on $\mathcal{O}(C)$ is a valuation and thus $|f|_{\sup} = 1$. We denote by $\bar{f}$ the image of f by the canonical reduction map $\mathcal{O}(C)^\circ \to \widetilde{\mathcal{O}(C)}$. Since $\bar{f} \in \widetilde{\mathcal{O}(C)}^\times$, by (10.2) we can write as

$$\bar{f} = \bar{\lambda} t^n$$

with some $\bar{\lambda} \in k^\times$ and $n \in \mathbb{Z}$. Take a lift $\lambda' \in \mathcal{O}_{\mathbb{K}}^\times$ of $\bar{\lambda}$. Then we have

$$f(\lambda' t^n)^{-1} = 1 + \mathcal{O}(C)^\vee$$

and thus we can write f as in the lemma. □

Definition 10.7. Let (C, ψ) be an oriented circle and let t be its positive parameter. Let ω be a holomorphic differential form on C. Write

$$\omega = \sum_{n \in \mathbb{Z}} a_n t^n dt, \quad a_n \in \mathbb{K}.$$

Then we define

$$\mathrm{Res}_t(\omega) = a_{-1}$$

and call it the residue of ω with respect to C (or (C, ψ)).

Let $\Omega^1_{C/\mathbb{K}}$ be the module of rigid analytic differential forms on C. We can write $\Omega^1_{C/\mathbb{K}} = \mathcal{O}(C)dt$ and the topology on it induced by the Banach topology of $\mathcal{O}(C)$ is independent of the choice of a parameter t [BGR, Proposition 3.7.3/3]. By Definition 10.4 (3), the map

$$\mathrm{Res}_t : \Omega^1_{C/\mathbb{K}} \to \mathbb{K}$$

is continuous (with respect to the Banach norm on $\mathcal{O}(C)$). Moreover, since

$$\frac{dt^{-1}}{t^{-1}} = -t\frac{dt}{t^2} = -\frac{dt}{t},$$

we have $\mathrm{Res}_{t^{-1}}(\omega) = -\mathrm{Res}_t(\omega)$, where the former residue is with respect to $(C, \bar{\psi})$.

Lemma 10.8. $\mathrm{Res}_t(\omega)$ *does not depend on the choice of a positive parameter* t.

Proof. Let s be another positive parameter of (C, ψ). By the continuity of Res_s, it is enough to show

$$\mathrm{Res}_s(t^m \frac{dt}{t}) = \begin{cases} 1 & (m = 0) \\ 0 & (m \neq 0). \end{cases}$$

For $m < 0$, we have

$$t^m \frac{dt}{t} = -(t^{-1})^{-m} \frac{d(t^{-1})}{t^{-1}}$$

and thus

$$\mathrm{Res}_s(t^m \frac{dt}{t}) = -\mathrm{Res}_s\left((t^{-1})^{-m} \frac{d(t^{-1})}{t^{-1}}\right) = \mathrm{Res}_{s^{-1}}\left((t^{-1})^{-m} \frac{d(t^{-1})}{t^{-1}}\right).$$

Since both of s^{-1} and t^{-1} are positive for the opposite orientation $\bar{\psi}$, we may assume $m \geqslant 0$.

By Lemma 10.6, we can write

$$t = \lambda s(1 + \sum_{n \neq 0} a_n s^n)$$

with $\lambda, a_n \in \mathbb{K}$ satisfying $|\lambda| = 1$ and $|a_n| < 1$ for any n. Again by the continuity of Res_s, we may assume that there are only finitely many nonzero a_n. Then for some non-negative integers l, l' we have

$$t = \lambda s \cdot s^{-l} f(s), \quad f(s) = \sum_{i=0}^{l-1} a_{i-l} s^i + s^l + \sum_{j=l+1}^{l+l'} a_{j-l} s^j$$

with $a_{-l} \neq 0$ and $a_{l'} \neq 0$. Here we consider the sum is zero when $l = 0$ or $l' = 0$.

Suppose $m = 0$. Then inspecting the Newton polygon shows that the polynomial $f(s)$ has exactly l roots $\alpha_1, \ldots, \alpha_l$ with absolute value less than one and exactly l' roots $\beta_1, \ldots, \beta_{l'}$ with absolute value more than one. Moreover, inspecting the right endpoint of the Newton polygon we obtain $|a_{l'}\beta_1 \cdots \beta_{l'}| = 1$ when $l' > 0$. Thus we can write

$$t = s\mu \prod_{i=1}^{l} (1 - \alpha_i s^{-1}) \prod_{j=1}^{l'} (1 - \beta_j^{-1} s)$$

with some $\mu \in \mathbb{K}$ satisfying $|\mu| = 1$. Then

$$\frac{dt}{t} = \left(1 + \sum_{i=1}^{l} \frac{\alpha_i s^{-1}}{1 - \alpha_i s^{-1}} - \sum_{j=1}^{l'} \frac{\beta_j^{-1} s}{1 - \beta_j^{-1} s}\right) \frac{ds}{s},$$

which yields $\mathrm{Res}_s(\frac{dt}{t}) = 1$.

Suppose $m > 0$. If $\mathrm{char}(\mathbb{K}) = 0$, then writing $t^m = \sum_{n \in \mathbb{Z}} b_n s^n$ we have

$$t^m \frac{dt}{t} = \frac{1}{m} d(t^m) = \frac{1}{m} \left(\sum_{n \in \mathbb{Z}} n b_n s^n\right) \frac{ds}{s},$$

which yields $\mathrm{Res}_s(t^m \frac{dt}{t}) = 0$.

If $\mathrm{char}(\mathbb{K}) > 0$, write

$$t^m \frac{dt}{t} = \lambda^m s^m \left(1 + \sum_{n \neq 0} a_n s^n\right)^{m-1} \left(1 + \sum_{n \neq 0} (n+1) a_n s^n\right) \frac{ds}{s}.$$

Since $m > 0$, there exists a polynomial $P \in \mathbb{Z}[X_{-l}, \ldots, X_{l'}]$ such that the constant term of the Laurent polynomial

$$s^m \left(1 + \sum_{n \neq 0} a_n s^n\right)^{m-1} (1 + \sum_{n \neq 0} (n+1) a_n s^n)$$

in s is $P(a_{-l}, \ldots, a_{l'})$.

Consider the fraction field of the p-adic completion of the localization of $\mathbb{Z}[X_{-l}, \ldots, X_{l'}]$ at (p). Let L be the p-adic completion of its algebraic closure. Since L is algebraically closed with $\mathrm{char}(L) = 0$, we have $\mathrm{Res}_{s_0}(t^m \frac{dt}{t}) = 0$ with a parameter s_0 and

$$t = s_0 \cdot s_0^{-l} f(s_0), \quad f(s_0) = \sum_{i=0}^{l-1} X_{i-l} s_0^i + s_0^l + \sum_{j=l+1}^{l+l'} X_{j-l} s_0^j.$$

This implies $P(X_{-l}, \ldots, X_{l'}) = 0$ and $P(a_{-l}, \ldots, a_{l'}) = 0$. □

Definition 10.9. Let (C, ψ) be an oriented circle. Choose a positive parameter t of (C, ψ). For any $\omega \in \Omega^1_{C/\mathbb{K}}$, we define

$$\mathrm{Res}_{(C,\psi)}(\omega) := \mathrm{Res}_t(\omega).$$

We also denote it by $\mathrm{Res}_C(\omega)$ if there is no risk of confusion.

10.2. **Discs and orientations of boundary circles.**

Definition 10.10. For any $a \in \mathbb{K}$ and $\rho \in |\mathbb{K}^\times|$, let

$$D_{\mathbb{K}}(a, \rho) = \{z \in \mathbb{K} \mid |z-a| \leqslant \rho\}, \quad D'_{\mathbb{K}}(a, \rho) = \{z \in \mathbb{K} \mid |z-a| \geqslant \rho\} \cup \{\infty\}.$$

We call them closed discs in $\mathbb{P}^1_{\mathbb{K}}$ and we refer to a as a center of these closed discs. Moreover, we put

$$D^\circ_{\mathbb{K}}(a, \rho) = \{z \in \mathbb{K} \mid |z-a| < \rho\}, \quad D'^\circ_{\mathbb{K}}(a, \rho) = \{z \in \mathbb{K} \mid |z-a| > \rho\} \cup \{\infty\}$$

and call them the interiors of the closed discs. We also put

$$\partial D_{\mathbb{K}}(a, \rho) = \partial D'_{\mathbb{K}}(a, \rho) = \{z \in \mathbb{K} \mid |z - a| = \rho\},$$

which we call the boundary of the closed discs. Then the latter is a circle. For any closed disc D in $\mathbb{P}^1_{\mathbb{K}}$, we denote by D° its interior and by ∂D its boundary.

By Lemma 4.3, there exists a unique $\rho \in |\mathbb{K}^\times|$ satisfying $D = D_{\mathbb{K}}(a, \rho)$ or $D = D'_{\mathbb{K}}(a, \rho)$ with some $a \in \mathbb{K}$, while such $a \in \mathbb{K}$ is not unique. Note that D° and ∂D depend on the choice of a center a of D. Thus we also write D° as $\mathrm{Int}_a D$ and ∂D as $\partial_a D$.

Example 10.11. For $D := D_{\mathbb{K}}(0,1) = D_{\mathbb{K}}(1,1)$, we have $D^{\circ}_{\mathbb{K}}(0,1) = m_{\mathbb{K}}$, $D^{\circ}_{\mathbb{K}}(1,1) = 1 + m_{\mathbb{K}}$ and

$$D^{\circ}_{\mathbb{K}}(0,1) \cap D^{\circ}_{\mathbb{K}}(1,1) = \varnothing, \quad \partial_0 D = \mathcal{O}_{\mathbb{K}}^{\times} \neq 1 + \mathcal{O}_{\mathbb{K}}^{\times} = \partial_1 D.$$

Definition 10.12. Let D be a closed disc in $\mathbb{P}^1_{\mathbb{K}}$. Put

$$t_D = \begin{cases} \frac{x-a}{\varpi_\rho} & (D = D_{\mathbb{K}}(a,\rho)) \\ \frac{\varpi_\rho}{x-a} & (D = D'_{\mathbb{K}}(a,\rho)). \end{cases}$$

We call t_D the standard parameter of the closed disc D. Then t_D is a parameter of the circle ∂D, and defines an orientation

$$\psi_D : \mathbb{Z} \to \widetilde{\mathcal{O}(\partial D)}^{\times}/k^{\times}, \quad 1 \mapsto t_D.$$

We call ψ_D the orientation of the circle ∂D associated with D. If we write

$$\{D, D'\} = \{D_{\mathbb{K}}(a,\rho), D'_{\mathbb{K}}(a,\rho)\},$$

then the orientations ψ_D and $\psi_{D'}$ of the circle $\partial D = \partial D'$ are opposite to each other.

Note that we have an isometric isomorphism of affinoid algebras over $\mathbb{K}$

$$\mathbb{K}\langle x\rangle \to \mathcal{O}(D), \quad x \mapsto t_D,$$

by which we often identify these affinoid algebras.

Lemma 10.13. *Let D and D' be closed discs in $\mathbb{P}^1_{\mathbb{K}}$ with center a and a', respectively. Let $f : D \to D'$ be an isomorphism of rigid analytic varieties over $\mathbb{K}$ such that f induces an isomorphism $\partial_a D \to \partial_{a'} D'$. Let $\omega \in \Omega^1_{\partial_{a'} D'/\mathbb{K}}$. Then we have*

$$\mathrm{Res}_{(\partial_a D, \psi_D)}(f^*\omega) = \mathrm{Res}_{(\partial_{a'} D', \psi_{D'})}(\omega).$$

Proof. Put $D_0 = D_{\mathbb{K}}(0,1) = \mathrm{Sp}(\mathbb{K}\langle x\rangle)$. Consider the isomorphisms

$$g : D_0 \to D_1 := D_{\mathbb{K}}(a,\rho), \quad z \mapsto \frac{z-a}{\varpi},$$
$$g' : D_0 \to D'_1 := D'_{\mathbb{K}}(a,\rho), \quad z \mapsto \frac{\varpi}{z-a}$$

with $\varpi \in \mathbb{K}^{\times}$ satisfying $|\varpi| = \rho$. Then they induce isomorphisms

$$\partial_0 D_0 \to \partial_a D_1, \quad \partial_0 D_0 \to \partial_a D'_1.$$

Moreover, if we write $\omega \in \Omega^1_{\partial_a D_1/\mathbb{K}}$ as

$$\omega = \sum_{n \in \mathbb{Z}} a_n \left(\frac{x-a}{\varpi}\right)^n d\left(\frac{x-a}{\varpi}\right),$$

Then $g^*\omega$ equals

$$g^*\omega = \sum_{n\in\mathbb{Z}} a_n x^n dx.$$

and similarly for D_1'. By the definition of circular residues, this implies

$$\begin{aligned}\mathrm{Res}_{(\partial_0 D_0,\psi_{D_0})}(g^*\omega) &= \mathrm{Res}_{(\partial_a D_1,\psi_{D_1})}(\omega),\\ \mathrm{Res}_{(\partial_0 D_0,\psi_{D_0})}((g')^*\omega') &= \mathrm{Res}_{(\partial_a D_1',\psi_{D_1'})}(\omega')\end{aligned}$$

for any $\omega \in \Omega^1_{\partial_a D_1/\mathbb{K}}$ and $\omega' \in \Omega^1_{\partial_a D_1'/\mathbb{K}}$. Hence, by composing these isomorphisms with f, we may assume $D = D' = D_0$ and $a = a' = 0$.

Consider the isomorphisms of affinoid algebras over $\mathbb{K}$

$$f^* : \mathbb{K}\langle x\rangle \to \mathbb{K}\langle x\rangle, \quad g^* : \mathbb{K}\langle x, x^{-1}\rangle \to \mathbb{K}\langle x, x^{-1}\rangle$$

which f induces. Put $F = f^*(x)$. Then F is invertible in the ring $\mathbb{K}\langle x, x^{-1}\rangle$ and any element G of $\mathbb{K}\langle x, x^{-1}\rangle$ can be uniquely written as

$$G = \sum_{n\in\mathbb{Z}} a_n F^n, \quad \lim_{n\to\infty} a_n = \lim_{n\to-\infty} a_n = 0.$$

By [BGR, Proposition 6.2.2/1], these maps are isometric with respect to the supremum norm. Thus we have

$$|F|_{\sup} = |x|_{\sup} = 1, \quad |G|_{\sup} = |\sum_{n\in\mathbb{Z}} a_n x^n|_{\sup} = \max\{|a_n| \mid n \in \mathbb{Z}\}.$$

Hence F is a parameter of the circle $\partial_0 D_0$.

Moreover, the map f induces isomorphisms of k-algebras

$$\tilde{f}^* : \widetilde{\mathbb{K}\langle x\rangle} = k[x] \to k[x], \quad \tilde{g}^* : \widetilde{\mathbb{K}\langle x, x^{-1}\rangle} = k[x, x^{-1}] \to k[x, x^{-1}].$$

This shows that $\tilde{f}^*(x) = \bar{a}x + \bar{b}$ with some $\bar{a}, \bar{b} \in k$ satisfying $\bar{a} \neq 0$. Since it is invertible in the ring $k[x, x^{-1}]$, we have $\bar{b} = 0$. This implies that the parameter F is positive for the orientation ψ_{D_0} of $\partial_0 D_0$.

For any $\omega \in \Omega^1_{\partial_0 D_0/\mathbb{K}}$, write

$$\omega = \sum_{n\in\mathbb{Z}} a_n x^n dx, \quad f^*\omega = \sum_{n\in\mathbb{Z}} a_n F^n dF.$$

By Lemma 10.8, we obtain

$$\mathrm{Res}_{(\partial_0 D_0,\psi_{D_0})}(f^*\omega) = \mathrm{Res}_F(f^*\omega) = a_{-1} = \mathrm{Res}_{(\partial_0 D_0,\psi_{D_0})}(\omega).$$

This concludes the proof. □

10.3. **Rigid analytic residue theorem on discs.** Let Y be a connected affinoid admissible open subset of $\mathbb{P}^1_{\mathbb{K}}$. By the maximal modulus principle and [BGR, Corollary 8.2.1/4], we see that Y is an affinoid subdomain of a closed disc. Then Lemma 10.1 implies that $\mathcal{O}(Y)$ is a PID. We denote by $K(Y)$ the fraction field of $\mathcal{O}(Y)$.

Definition 10.14. For any connected affinoid admissible open subset Y of $\mathbb{P}^1_{\mathbb{K}}$, we call any element of $K(Y) \otimes_{\mathcal{O}(Y)} \Omega^1_{Y/\mathbb{K}}$ a meromorphic differential form on Y.

Definition 10.15. Let Y be a connected affinoid admissible open subset of $\mathbb{P}^1_{\mathbb{K}}$ and $c \in Y$. Put $t = x - c$ when $c \neq \infty$ and $t = \frac{1}{x}$ when $c = \infty$. Let ω be a meromorphic differential form on Y. Since $\mathcal{O}(Y)$ is a PID, we can write at c

$$\omega = \sum_{n \geqslant -N} a_n t^n dt$$

with some $a_n \in \mathbb{K}$ and $N \in \mathbb{Z}$. Then we define $\mathrm{Res}_c(\omega) = a_{-1}$.

Lemma 10.16. *Let Y be a connected affinoid admissible open subset of $\mathbb{P}^1_{\mathbb{K}}$ and $c \in Y$. Put t be as in Definition 10.15. Let $m \geqslant 0$ be an integer. Then the map*

$$\mathcal{O}(Y) \to \mathbb{K}, \quad f \mapsto \mathrm{Res}_c(t^{-m} f dt)$$

is continuous with respect to the Banach norm on $\mathcal{O}(Y)$.

Proof. Take a closed disc $D = D_{\mathbb{K}}(c, \rho)$ contained in Y. Since the map $\mathcal{O}(Y) \to \mathcal{O}(D)$ is continuous and the map of the lemma factors through this map, we may assume $Y = D$. Take any $\varpi \in \mathbb{K}$ satisfying $|\varpi| = \rho$. Then $f \in \mathcal{O}(D)$ is written as

$$f = \sum_{n \geqslant 0} a_n \left(\frac{t}{\varpi}\right)^n, \quad \lim_{n \to \infty} a_n = 0$$

and the Banach norm of $\mathcal{O}(D)$ is given by $|f| = \max\{|a_n| \mid n \geqslant 0\}$. Since $\mathrm{Res}_c(t^{-m} f dt) = \frac{a_{m-1}}{\varpi^{m-1}}$, the continuity follows. □

Lemma 10.17 (Rigid analytic residue theorem)**.** *Let D be a closed disc in $\mathbb{P}^1_{\mathbb{K}}$. We consider ∂D as an oriented circle by the orientation ψ_D associated with D. Let ω be a meromorphic differential form on D which has no poles on ∂D. Then we have*

$$\mathrm{Res}_{\partial D}(\omega) = \sum_{c \in D^\circ} \mathrm{Res}_c(\omega).$$

Proof. Write $\omega = \frac{f}{g} dt_D$ with $f, g \in \mathcal{O}(D)$. Since $\mathcal{O}(D)$ is a PID, we may assume that f and g are coprime and

$$g = h \prod_{i=1}^{r} (t_D - \alpha_i)^{n_i}$$

with some $\alpha_i \in \mathbb{K}$ satisfying $|\alpha_i| < 1$, $n_i \in \mathbb{Z}_{>0}$ and $h \in \mathcal{O}(D)^\times$. By the Weierstrass division theorem [BGR, Theorem 5.2.1/2], we can write

$$fh^{-1} = Q \prod_{i=1}^{r} (t_D - \alpha_i)^{n_i} + R$$

with some $Q \in \mathcal{O}(D)$ and $R \in \mathbb{K}[t_D]$ with $\deg(R) < \sum_{i=1}^{r} n_i$. Thus, by the partial fraction decomposition we can write

$$\omega = \left(\sum_{i=1}^{r} \sum_{n=1}^{n_i} \frac{a_{i,n}}{(t_D - \alpha_i)^n} + \sum_{m \geqslant 0} b_m t_D^m \right) dt_D.$$

From the equality

$$\frac{a_{i,n}}{(t_D - \alpha_i)^n} = \frac{a_{i,n}}{t_D^n} \cdot \frac{1}{(1 - \frac{\alpha_i}{t_D})^n} = \frac{a_{i,n}}{t_D^n} \sum_{j \geqslant 0} \binom{-n}{j} (\frac{-\alpha_i}{t_D})^j,$$

we obtain $\mathrm{Res}_{\partial D}(\omega) = \sum_{i=1}^{r} a_{i,1}$.

Let us compute $\mathrm{Res}_c(\omega)$ for any $c \in D^\circ$. When $D = D_{\mathbb{K}}(a, \rho)$ and $t_D = \frac{x-a}{\varpi_\rho}$, we have

$$dt_D = \frac{dx}{\varpi_\rho} = \frac{d(x-c)}{\varpi_\rho}, \quad t_D - \alpha_i = \frac{x - (a + \varpi_\rho \alpha_i)}{\varpi_\rho}.$$

Thus $\mathrm{Res}_c(\omega) = 0$ unless $c = a + \varpi_\rho \alpha_i$ for some $i = 1, \ldots, r$. When the latter equality holds, then $\mathrm{Res}_c(\omega) = a_{i,1}$ and the lemma follows for this case.

Suppose $D = D'_{\mathbb{K}}(a, \rho)$ and $t_D = \frac{\varpi_\rho}{x-a}$. Then $a \neq c$. For $c \neq \infty$, we have

$$t_D = \frac{\varpi_\rho}{x - c + (c - a)} = \frac{\varpi_\rho}{c - a} \sum_{j \geqslant 0} \frac{(-1)^j}{(c-a)^j} (x - c)^j,$$

$$dt_D = \frac{\varpi_\rho}{c - a} \sum_{j \geqslant 0} \frac{j(-1)^j}{(c-a)^j} (x - c)^{j-1} d(x - c),$$

$$t_D - \alpha_i = \frac{-(\alpha_i x - (a\alpha_i + \varpi_\rho))}{x - a}.$$

Thus $\mathrm{Res}_c(\omega) = 0$ unless $c = a + \varpi_\rho \alpha_i^{-1}$ for some $i = 1, \ldots, r$ satisfying $\alpha_i \neq 0$. In the latter case, we have

$$
\begin{aligned}
t_D - \alpha_i &= \frac{-\alpha_i(x-c)}{x - c + (c-a)}, \\
\frac{a_{i,n}}{(t_D - \alpha_i)^n} dt_D &= \frac{a_{i,n}}{(-\alpha_i)^n} \sum_{l=0}^{n} \binom{n}{l} (c-a)^{n-l} \\
&\quad \cdot \frac{\varpi_\rho}{c-a} \sum_{j \geqslant 0} \frac{j(-1)^j}{(c-a)^j} (x-c)^{j+l-1-n} d(x-c).
\end{aligned}
$$

Its residue at c only comes from the term of $j = n-l$. For $j = n-l$ and $l = n$, we have $j = 0$ and the residue vanishes. Hence, by $c - a = \varpi_\rho \alpha_i^{-1}$ the residue equals

$$
\frac{na_{i,n}}{-(-\alpha_i)^{n-1}} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^{n-l} = \left\{ \begin{array}{ll} 0 & (n > 1), \\ a_{i,1} & (n = 1). \end{array} \right.
$$

For $c = \infty$, we have

$$
t_D = \frac{1}{x} \cdot \frac{\varpi_\rho}{1 - \frac{a}{x}} = \varpi_\rho \sum_{j \geqslant 0} \frac{a^j}{x^{j+1}}, \quad dt_D = \varpi_\rho \sum_{j \geqslant 0} (j+1) \frac{a^j}{x^j} d(\frac{1}{x}).
$$

If $\alpha_i \neq 0$, then

$$
t_D - \alpha_i = -\alpha_i \left(1 - \frac{\varpi_\rho}{\alpha_i} \sum_{j \geqslant 0} \frac{a^j}{x^{j+1}} \right)
$$

and thus $\mathrm{Res}_\infty(\omega) = 0$. If $\alpha_i = 0$, then we have

$$
\begin{aligned}
\frac{a_{i,n}}{t_D^n} dt_D &= a_{i,n} \left(\frac{x-a}{\varpi_\rho} \right)^n \varpi_\rho \sum_{j \geqslant 0} (j+1) \frac{a^j}{x^j} d(\frac{1}{x}) \\
&= \frac{a_{i,n} x^n}{\varpi_\rho^n} \left(1 - \frac{a}{x} \right)^n \varpi_\rho \sum_{j \geqslant 0} (j+1) \frac{a^j}{x^j} d(\frac{1}{x}) \\
&= \frac{a_{i,n} x^n}{\varpi_\rho^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{a^l}{x^l} \varpi_\rho \sum_{j \geqslant 0} (j+1) \frac{a^j}{x^j} d(\frac{1}{x}).
\end{aligned}
$$

Then its residue at ∞ only comes from the term of $j = n-1-l$. Thus the residue equals

$$
\frac{na_{i,n} a^{n-1}}{\varpi_\rho^{n-1}} \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} = \left\{ \begin{array}{ll} 0 & (n > 1), \\ a_{i,1} & (n = 1). \end{array} \right.
$$

Hence the lemma follows also for this case. □

10.4. **Connected affinoids in $\mathbb{P}^1_{\mathbb{K}}$.** .

Let $I = \{D_1, \ldots, D_r\}$ be a nonempty finite set of closed discs in $\mathbb{P}^1_{\mathbb{K}}$. Let $a_i \in \mathbb{K}$ be a center of D_i and let $D_i^\circ = \mathrm{Int}_{a_i} D_i$. Suppose either

(1) $D_1^\circ, \ldots, D_r^\circ$ are disjoint to each other and $\infty \in D_i^\circ$ for some i, or
(2) $D_1, \ldots, D_r$ are disjoint to each other.

Put

$$F_I = \mathbb{P}^1_{\mathbb{K}} \backslash \bigcup_{i=1}^r D_i^\circ, \quad F_I^\circ = \mathbb{P}^1_{\mathbb{K}} \backslash \bigcup_{i=1}^r D_i.$$

Then they are admissible open subsets of $\mathbb{P}^1_{\mathbb{K}}$. Since

$$F_I = \bigcap_{i=1}^r (\mathbb{P}^1_{\mathbb{K}} \backslash D_i^\circ)$$

is a finite intersection of closed discs in $\mathbb{P}^1_{\mathbb{K}}$, we see that F_I is an affinoid variety over $\mathbb{K}$. Moreover, since $\mathbb{P}^1_{\mathbb{K}}$ is reduced so is F_I, and by [BGR, Theorem 6.2.4/1] the supremum semi-norm on F_I is a complete norm which defines the Banach topology on $\mathcal{O}(F_I)$.

Lemma 10.18. *Let $I = \{D_1, \ldots, D_r\}$ and $a_i \in \mathbb{K}$ be as above. Then there exists $\gamma \in GL_2(\mathbb{K})$ satisfying the following conditions.*

- *$\gamma(D_1) = D'_{\mathbb{K}}(0,1)$.*
- *$\gamma(D_i)$ is a closed disc in $\mathbb{P}^1_{\mathbb{K}}$ for any i.*
- *With some choice of centers of $\gamma(D_i)$, the interiors $\gamma(D_i)^\circ$ are disjoint to each other.*

Proof. First suppose $\infty \notin F_I$. We may assume $D_1 = D'_{\mathbb{K}}(a_1, \rho_1)$ and $D_i = D_{\mathbb{K}}(a_i, \rho_i)$ for $i \neq 1$ with some $\rho_i \in |\mathbb{K}^\times|$. Take any $\varpi_i \in \mathbb{K}$ satisfying $|\varpi_i| = \rho_i$ and put $\gamma = \begin{pmatrix} 1 & -a_1 \\ 0 & \varpi_1 \end{pmatrix}$. Then $\gamma(D_1) = D'_{\mathbb{K}}(0,1)$ and $\gamma(D_1^\circ) = D'^\circ_{\mathbb{K}}(0,1)$. For any $i \neq 1$, we have $|0 \cdot a_i + \varpi_1| = \rho_1 > 0 = \rho_i |0|$. By Lemma 4.8 and the complement of the former equality of Lemma 4.9, we obtain

$$\gamma(D_{\mathbb{K}}(a_i, \rho_i)) = D_{\mathbb{K}}\left(\gamma(a_i), \frac{\rho_i}{\rho_1}\right), \quad \gamma(D^\circ_{\mathbb{K}}(a_i, \rho_i)) = D^\circ_{\mathbb{K}}\left(\gamma(a_i), \frac{\rho_i}{\rho_1}\right).$$

Since $D_1^\circ, \ldots, D_r^\circ$ are disjoint, so are $\gamma(D_1^\circ), \ldots, \gamma(D_r^\circ)$ and the lemma holds for $\mathrm{Int}_{\gamma(a_i)} \gamma(D_i) = \gamma(D_i^\circ)$.

Next suppose $\infty \in F_I$, so that the assumption (2) holds. For any $i = 1, \ldots, r$, we can write $D_i = D_{\mathbb{K}}(a_i, \rho_i)$ with some $\rho_i \in |\mathbb{K}^\times|$ satisfying $|a_i - a_j| > \max\{\rho_i, \rho_j\}$. Take any $\varpi_i \in \mathbb{K}$ satisfying $|\varpi_i| = \rho_i$ and put $\gamma = \begin{pmatrix} 0 & \varpi_1 \\ 1 & -a_1 \end{pmatrix}$. Then $\gamma(D_1) = D'_{\mathbb{K}}(0,1)$ and $\gamma(D_1^\circ) = D'^\circ_{\mathbb{K}}(0,1)$. For any $i \neq 1$, we have $|1 \cdot a_i - a_1| = |a_i - a_1| > \rho_i = \rho_i|1|$. By Lemma 4.8 and the complement of the former equality of Lemma 4.9, we obtain

$$\gamma(D_{\mathbb{K}}(a_i, \rho_i)) = D_{\mathbb{K}}\left(\gamma(a_i), \frac{\rho_1\rho_i}{|a_1 - a_i|^2}\right), \quad \gamma(D^{\circ}_{\mathbb{K}}(a_i, \rho_i)) = D^{\circ}_{\mathbb{K}}\left(\gamma(a_i), \frac{\rho_1\rho_i}{|a_1 - a_i|^2}\right).$$

Then the lemma holds similarly for $\mathrm{Int}_{\gamma(a_i)}\gamma(D_i) = \gamma(D_i^{\circ})$. $\square$

Lemma 10.19. *There exists a finite covering $\{Y_\lambda\}_{\lambda\in\Lambda}$ of F_I satisfying the following conditions.*

(1) Each Y_λ is an affinoid subdomain of F_I which is isomorphic to

$$F_0 := \{z \in \mathbb{K} \mid \rho \leqslant |z| \leqslant 1\}\backslash\bigcup_{i=1}^{m}\{z \in \mathbb{K} \mid |z - c_i| < 1\}$$

with some $\rho < 1$, $m \geqslant 0$ and $c_i \in \mathbb{K}$ satisfying $|c_i| = 1$.

(2) For any $\lambda, \lambda' \in \Lambda$, there exist $\lambda_1, \ldots, \lambda_N \in \Lambda$ such that $\lambda = \lambda_1$, $\lambda' = \lambda_N$ and $Y_{\lambda_i} \cap Y_{\lambda_{i+1}} \neq \varnothing$ for any $i = 1, \ldots, N-1$.

Proof. By Lemma 10.18, we may assume $D_1 = D'_{\mathbb{K}}(0, 1)$. Then for any $i = 2, \ldots, r$, we can write

$$D_i = D_{\mathbb{K}}(a_i, \rho_i), \quad |a_i| \leqslant 1, \ \rho_i \leqslant 1,$$

where we have $|a_i - a_j| \geqslant \rho_i$ for any $i, j \in \{2, \ldots, r\}$ satisfying $i \neq j$. Then

$$F_I = \{z \in \mathbb{K} \mid |z| \leqslant 1, \ |z - a_i| \geqslant \rho_i \text{ for all } i = 2, \ldots, r\}.$$

Put

$$F_i = \{z \in F_I \mid |z - a_i| \leqslant |z - a_j| \text{ for all } j = 2, \ldots, r\},$$

which is a rational subdomain of F_I satisfying $F_I = \bigcup_{i=2}^{r} F_i$.

Fix some $i = 2, \ldots, r$. Note that we have $\rho_i \leqslant |a_i - a_j| \leqslant 1$ for any $j \neq i$. Let

$$\rho_i = r_1 < r_2 < \cdots < r_s < r_{s+1} = 1$$

be the elements of the set $\{\rho_i, 1\} \cup \{|a_i - a_j| \mid j \neq i\}$. Put

$$J_t = \{j = 2, \ldots, r \mid |a_i - a_j| = r_t\}.$$

Take some $r'_i \in |\mathbb{K}^{\times}|$ satisfying $r_i < r'_i < r_{i+1}$. Define

$$X_t = \{z \in \mathbb{K} \mid r_t \leqslant |z - a_i| \leqslant r'_t\}\backslash\bigcup_{j\in J_t}\{z \in \mathbb{K} \mid |z - a_j| < r_t\},$$

$$X'_t = \{z \in \mathbb{K} \mid r'_t \leqslant |z - a_i| \leqslant r_{t+1}\}\backslash\bigcup_{j\in J_{t+1}}\{z \in \mathbb{K} \mid |z - a_j| < r_{t+1}\}.$$

Then we claim $F_i = \bigcup_{t=1}^{s}(X_t \cup X'_t)$.

Take any $z \in X_t$. Then $|z - a_i| \geqslant r_t \geqslant \rho_i$. Since $|z - a_i| \leqslant r'_t \leqslant 1$ and $|a_i| \leqslant 1$, we have $|z| \leqslant 1$. For any $j \neq i$ satisfying $j \notin J_t$, we have $|a_i - a_j| \neq r_t$ and $r_t \leqslant |z - a_i| \leqslant r'_t$. Hence $|z - a_i| \neq |a_i - a_j|$ and

$$|z - a_j| = \max\{|z - a_i|, |a_j - a_i|\},$$

which yields $|z - a_j| \geqslant |z - a_i|$ and $|z - a_j| \geqslant |a_j - a_i| \geqslant \rho_j$. For any $j \in J_t$, we have $|z - a_j| \geqslant r_t = |a_i - a_j| \geqslant \rho_j$. If $|z - a_i| > |z - a_j|$, then $r_t = |a_i - a_j| = |z - a_i| > |z - a_j|$, which is a contradiction. Hence we obtain $|z - a_i| \leqslant |z - a_j|$ and $z \in F_i$.

Take any $z \in X'_t$. Then $|z - a_i| \geqslant r'_t \geqslant \rho_i$. Since $|z - a_i| \leqslant r_{t+1} \leqslant 1$ and $|a_i| \leqslant 1$, we have $|z| \leqslant 1$. For any $j \neq i$ satisfying $j \notin J_{t+1}$, we have $|a_i - a_j| \neq r_{t+1}$ and $r'_t \leqslant |z - a_i| \leqslant r_{t+1}$. Hence $|z - a_i| \neq |a_i - a_j|$ and

$$|z - a_j| = \max\{|z - a_i|, |a_j - a_i|\},$$

which yields $|z - a_j| \geqslant |z - a_i|$ and $|z - a_j| \geqslant |a_j - a_i| \geqslant \rho_j$. For any $j \in J_{t+1}$, we have $|z - a_j| \geqslant r_{t+1} = |a_i - a_j| \geqslant \rho_j$. If $|z - a_i| > |z - a_j|$, then $r_{t+1} = |a_i - a_j| = |z - a_i| > |z - a_j|$, which is a contradiction. Hence we obtain $|z - a_i| \leqslant |z - a_j|$ and $z \in F_i$.

On the other hand, take any $z \in F_i$. Since $|z| \leqslant 1$ and $|a_j| \leqslant 1$, we have $|z - a_i| \leqslant |z - a_j| \leqslant 1$ for any j.

Suppose $r_t < |z - a_i| < r_{t+1}$ with some $t = 1, \ldots, s$. For any $j \in J_t$, we have $r_t = |a_j - a_i|$ and $|z - a_j| = |z - a_i| > r_t$. Similarly, for any $j \in J_{t+1}$, we have $r_{t+1} = |a_j - a_i|$ and $|z - a_j| = r_{t+1}$. This implies $z \in X_t$ if $|z - a_i| \leqslant r'_t$ and $z \in X'_t$ otherwise.

Suppose $|z - a_i| = 1$. Then we have $1 = |z - a_i| \leqslant |z - a_j| \leqslant 1$ and $|z - a_j| = 1$ for any j. This yields $z \in X'_s$.

Suppose $|z - a_i| = r_t$ with some $t = 1, \ldots, s$. For any $j \in J_t$, we have $|z - a_j| = |z - a_i + (a_i - a_j)| \leqslant r_t = |z - a_i|$. Since $z \in F_i$, the inequality $|z - a_j| \geqslant |z - a_i|$ forces $|z - a_j| = |z - a_i| = r_t$. Hence we obtain $z \in X_t$. This concludes the proof of the claim.

Since X_t and X'_t are rational subdomains of $D_{\mathbb{K}}(0, 1)$, [BGR, Proposition 7.2.2/4] implies that they are also affinoid subdomains of F_I. Take any $\varpi_t \in \mathbb{K}$ satisfying $|\varpi_t| = r_t$. For $j \in J_{t+1}$, write $a_j - a_i = \varpi_{t+1} c_j$ with $|c_j| = 1$. Then the map $z \mapsto \frac{z-a_i}{\varpi_{t+1}}$ induces an isomorphism

$$\{z \in \mathbb{K} \mid \frac{r'_t}{r_{t+1}} \leqslant |z| \leqslant 1\} \setminus \bigcup_{j \in J_{t+1}} \{z \in \mathbb{K} \mid |z - c_j| < 1\} \to X'_t.$$

On the other hand, for $j \in J_t$, write $a_j - a_i = \varpi_t u_j$ with $|u_j| = 1$. Then the map $z \mapsto \frac{\varpi_t}{z-a_i}$ gives an isomorphism

$$\{z \in \mathbb{K} \mid \frac{r_t}{r'_t} \leqslant |z| \leqslant 1\} \setminus \bigcup_{j \in J_t} \{z \in \mathbb{K} \mid |z^{-1} - u_j| < 1\} \to X_t.$$

For any $z \in \mathbb{K}$ satisfying $|z^{-1} - u_j| < 1$, we have $|z| = 1$ and thus $|z - u_j^{-1}| = |z||u_j^{-1}||z^{-1} - u_j| < 1$. Since the converse also holds, for any $j \in J_t$ we obtain

$$\{z \in \mathbb{K} \mid |z^{-1} - u_j| < 1\} = \{z \in \mathbb{K} \mid |z - u_j^{-1}| < 1\}$$

and thus the map above gives an isomorphism

$$\{z \in \mathbb{K} \mid \frac{r_t}{r'_t} \leqslant |z| \leqslant 1\} \backslash \bigcup_{j \in J_t} \{z \in \mathbb{K} \mid |z - u_j^{-1}| < 1\} \to X_t.$$

Hence the condition (1) holds.

Let us show the condition (2). Let $t \in \{1, \ldots, s\}$. Take any $z \in \mathbb{K}$ satisfying $|z - a_i| = r'_t$. For any $j \in J_t$, we have $|a_i - a_j| = r_t < r'_t$ and $|z - a_j| = |z - a_i| = r'_t > r_t$, which shows $z \in X_t$. For any $j \in J_{t+1}$, we have $|a_i - a_j| = r_{t+1} > r'_t$ and $|z - a_j| = r_{r+1}$, which shows $z \in X'_t$. Hence $X_t \cap X'_t \neq \varnothing$.

Let $t \in \{1, \ldots, s-1\}$. Since k is an infinite field, we can choose $u \in \mathbb{K}$ such that $|u| = 1$ and $|u - c_j| = 1$ for any $j \in J_{t+1}$. Then $z = a_i + \varpi_{t+1} u$ satisfies $|z - a_i| = |z - a_j| = r_{t+1}$ for any $j \in J_{t+1}$. Thus $z \in X'_t \cap X_{t+1}$ and $X'_t \cap X_{t+1} \neq \varnothing$.

Now we are reduced to showing $\bigcap_{i=2}^r F_i \neq \varnothing$. For this, fix some $i \in \{2, \ldots, r\}$. Since k is an infinite field, we can take $u \in \mathbb{K}$ satisfying $|u| = |u + a_i - a_j| = 1$ for all $j \in J_{s+1}$. Put $z = a_i + u$. Then we have $|z| \leqslant 1$, $|z - a_i| = |u| = 1$ and

$$|z - a_j| = |u + (a_i - a_j)| = 1 \quad \text{for any } j \neq i,$$

which yields $z \in \bigcap_{i=2}^r F_i$. This concludes the proof of the lemma. $\square$

Lemma 10.20. *Let $|-|$ be the Gauss norm on the Tate algebra $\mathbb{K}\langle x, y\rangle$. Let $f \in \mathcal{O}_{\mathbb{K}}\langle x, y\rangle$ satisfy $|f| = 1$ and let $\bar{f}$ be its image in $k[x, y]$ by the natural reduction map. Let $R = \mathbb{K}\langle x, y\rangle/(f)$ and $\bar{R} = k[x, y]/(\bar{f})$. Suppose that the rings R and $\bar{R}$ are reduced and $\mathrm{Spec}(\bar{R})$ is connected. Then the affinoid variety $\mathrm{Sp}(R)$ is connected.*

Proof. By [BGR, Proposition 9.1.4/8], it is enough to show that $\mathrm{Spec}(R)$ is connected.

For this, first we claim that the $\mathcal{O}_{\mathbb{K}}$-algebra

$$R_0 = \mathcal{O}_{\mathbb{K}}\langle x, y\rangle/(f)$$

is torsion free as an $\mathcal{O}_{\mathbb{K}}$-module. Suppose that we have $aF = fG$ with some $a \in \mathcal{O}_{\mathbb{K}} \backslash \{0\}$ and $F, G \in \mathcal{O}_{\mathbb{K}}\langle x, y\rangle$. Since the Gauss norm on $\mathbb{K}\langle x, y\rangle$ is a valuation, we have $|a||F| = |f||G| = |G|$. Since $|F| \leqslant 1$, we have $|G| \leqslant |a|$ and thus $G = aH$ with some $H \in \mathcal{O}_{\mathbb{K}}\langle x, y\rangle$, which yields $F = fH$ and the claim follows.

The claim implies that R_0 is a subalgebra of R. Hence R_0 agrees with the image of $\mathcal{O}_{\mathbb{K}}\langle x, y\rangle$ by the natural surjection

$$\mathbb{K}\langle x, y\rangle \to R.$$

Since R is reduced, [BGR, Proposition 6.2.1/4 (iii)] implies that the supremum semi-norm $|-|_{\sup}$ on R is a norm. Since the ring $\bar{R} = R_0 \otimes_{\mathcal{O}_{\mathbb{K}}} k$ is reduced, [BGR, Proposition 6.4.3/4] shows $R_0 = R^\circ$. By [BGR, Remark after Proposition 6.3.4/1], the ring R_0 is integrally closed in R.

Let $e \in R$ be an idempotent. Since $e^2 = e$, we have $e \in R_0$ and $|e|_{\sup} \leqslant 1$. Since $|e|_{\sup} = |e^2|_{\sup} \leqslant |e|_{\sup}^2$, we have $e = 0$ or $|e|_{\sup} = 1$. Suppose $e \neq 0$. Since $\mathrm{Spec}(\bar{R})$ is connected, there is no nontrivial idempotent in $\bar{R}$ and thus $|1 - e|_{\sup} < 1$. Since $(1-e)^2 = 1 - e$, this forces $e = 1$. □

Lemma 10.21. *The affinoid variety F_0 in Lemma 10.19 is connected.*

Proof. Let $\rho < 1$ and c_i be as in Lemma 10.19 (1). Take $\varpi \in \mathbb{K}$ satisfying $|\varpi| = \rho$. For any $i = 1, \ldots, m$, let

$$Y_i = \{z \in \mathcal{O}_{\mathbb{K}} \mid |z - c_i| = 1\} = \mathrm{Sp}(\mathbb{K}\langle x, y\rangle/(y(x - c_i) - 1)),$$
$$Y_0 = \{z \in \mathcal{O}_{\mathbb{K}} \mid \rho \leqslant |z|\} = \mathrm{Sp}(\mathbb{K}\langle x, y\rangle/(xy - \varpi)).$$

Then they are rational subdomains of $\mathrm{Sp}(\mathbb{K}\langle x\rangle)$ satisfying $F_0 = \bigcap_{i=0}^m Y_i$.

Since Lemma 10.20 shows that each Y_i is connected, we are reduced to showing $\bigcap_{i=0}^m Y_i \neq \varnothing$. For this, since k is an infinite field we can find $u \in \mathbb{K}$ satisfying $|u| = |u - c_i| = 1$ for all $i = 1, \ldots, m$. Then $u \in \bigcap_{i=0}^m Y_i$ and the lemma follows. □

Lemma 10.22. *The affinoid variety F_I is connected.*

Proof. Take a finite covering $\{Y_\lambda\}_{\lambda \in \Lambda}$ of F_I as in Lemma 10.19. Suppose that we have a nontrivial decomposition $F_I = U \sqcup V$ into the disjoint union of affinoid subdomains. By Lemma 10.21 each Y_λ is connected and it is contained in either of U or V. Put

$$\Lambda_U = \{\lambda \in \Lambda \mid Y_\lambda \subseteq U\}, \quad \Lambda_V = \{\lambda \in \Lambda \mid Y_\lambda \subseteq V\}.$$

Since U and V are nonempty, so are Λ_U and Λ_V. Take $\lambda \in \Lambda_U$ and $\lambda' \in \Lambda_V$. By Lemma 10.19 (2), we have a finite subset of Λ

$$\lambda = \lambda_1, \lambda_2, \ldots, \lambda_N = \lambda'$$

satisfying $Y_{\lambda_i} \cap Y_{\lambda_{i+1}} \neq \varnothing$. Then there exists i such that $\lambda_i \in \Lambda_U$ and $\lambda_{i+1} \in \Lambda_V$. Since $U \cap V = \varnothing$, this is a contradiction. □

Lemma 10.23. *The ring $\mathcal{O}(F_I)$ is a PID of dimension one. Moreover, it contains the ring $R(F_I)$ of rational functions on $\mathbb{P}^1_{\mathbb{K}}$ with no poles in F_I as a dense subring.*

Proof. By Lemma 10.18, we may assume $D_1 = D'_{\mathbb{K}}(0,1)$. Then for any $i = 2,\dots,r$, we can write

$$D_i = D_{\mathbb{K}}(a_i,\rho_i), \quad |a_i| \leqslant 1,\ \rho_i \leqslant 1,$$

where we have $|a_i - a_j| \geqslant \rho_i$ for any $i \neq j$. Take $\varpi_i \in \mathbb{K}$ satisfying $|\varpi_i| = \rho_i$. Then the affinoid algebra of F_I is written as

$$\mathcal{O}(F_I) = \mathbb{K}\langle x, \frac{\varpi_2}{x-a_2},\dots,\frac{\varpi_r}{x-a_r}\rangle.$$

By Lemma 10.1 and Lemma 10.22, the ring $\mathcal{O}(F_I)$ is a PID of dimension one. Since k is an infinite field, we can find infinitely many $u \in \mathcal{O}_{\mathbb{K}}^{\times}$ satisfying $|u - a_i| = 1$ for all $i \in \{2,\dots,r\}$. Thus the function x is non-constant on F_I.

We claim that the natural map

$$\mathbb{K}(x) \to K(F_I)$$

into the fraction field $K(F_I)$ of $\mathcal{O}(F_I)$ is injective. If not, we can find a nonzero element $P(x) \in \mathbb{K}[x]$ which is zero on F_I. Since $\mathcal{O}(F_I)$ is a domain, this implies that x is a constant on F_I, which is a contradiction.

Let $a \in \mathbb{K}\backslash F_I$. Then $x-a \in \mathcal{O}(F_I)^{\times}$ and $\frac{1}{x-a} \in \mathcal{O}(F_I)$. By the partial fraction decomposition, we obtain $R(F_I) \subseteq \mathcal{O}(F_I)$. For the density, we have

$$\mathbb{K}[x, \frac{\varpi_2}{x-a_2},\dots,\frac{\varpi_r}{x-a_r}] \subseteq R(F_I) \subseteq \mathcal{O}(F_I)$$

and the leftmost ring is dense in $\mathcal{O}(F_I)$. Thus $R(F_I)$ is also dense in $\mathcal{O}(F_I)$. □

Remark 10.24. Let r,s be integers satisfying $s \geqslant -r$. By Lemma 5.20, the affinoid variety $\Omega_{r,s}$ is an example of F_I. Then Proposition 5.24 and Lemma 10.23 give the following description of the ring $\mathcal{O}(\Omega)$: A function $f : \Omega \to \mathbb{C}_\infty$ lies in $\mathcal{O}(\Omega)$ if and only if for any $r,s \in \mathbb{Z}_{\geqslant 0}$, the restriction $f|_{\Omega_{r,s}}$ is the limit of a sequence $\{F_n(x)\}_{n\geqslant 0}$ with respect to the supremum norm on $\Omega_{r,s}$ such that $F_n(x) \in \mathbb{C}_\infty(x)$ has no poles on $\Omega_{r,s}$.

10.5. **Rigid analytic residue theorem on connected affinoids in $\mathbb{P}^1_{\mathbb{K}}$.** Let $I = \{D_1,\dots,D_r\}$ be a nonempty finite set of closed discs in $\mathbb{P}^1_{\mathbb{K}}$ which are disjoint to each other. By Lemma 10.22, the affinoid algebra $\mathcal{O}(F_I)$ is a PID. Thus we may consider the module $K(F_I) \otimes_{\mathcal{O}(F_I)} \Omega^1_{F_I/\mathbb{K}}$ of meromorphic differential forms on F_I.

Theorem 10.25. *Let $I = \{D_1,\dots,D_r\}$ be a nonempty finite set of closed discs in $\mathbb{P}^1_{\mathbb{K}}$ which are disjoint to each other. Let $a_i \in \mathbb{K}$ be a center of D_i. We equip the circle $\partial D_i = \partial_{a_i} D_i$ with the orientation ψ_{D_i}*

associated with D_i. Let ω be a meromorphic differential form on F_I which has no poles on ∂D_i for all i. Then we have

$$\sum_{c\in F_I^\circ}\mathrm{Res}_c(\omega)+\sum_{i=1}^r \mathrm{Res}_{\partial D_i}(\omega)=0.$$

Proof. Take any $a\in\mathbb{K}\backslash F_I$. Put $t=\frac{1}{x-a}\in\mathcal{O}(F_I)$ so that $\mathbb{C}_\infty(x)=\mathbb{C}_\infty(t)$. By the maximal modulus principle, we can find $\rho\in|\mathbb{K}^\times|$ satisfying $F_I\subseteq D:=\{t\in\mathbb{K}\mid |t|\leqslant\rho\}$.

For any $b\in\mathbb{K}$ and $\sigma\in|\mathbb{K}^\times|$, we have

$$\begin{aligned}|x-b|\leqslant\sigma \quad&\Leftrightarrow\quad |(a-b)t+1|\leqslant\sigma|t|,\\ x=\infty \text{ or } |x-b|\geqslant\sigma \quad&\Leftrightarrow\quad |(a-b)t+1|\geqslant\sigma|t|.\end{aligned}$$

Thus F_I is a rational subdomain of D.

We can write

$$\mathcal{O}(F_I)=\mathcal{O}(D)\langle y_1,\ldots,y_r\rangle/(a_1y_1-b_1,\ldots,a_ry_r-b_r)$$

with some $a_i,b_i\in\mathcal{O}(D)$. Note that the relation $a_iy_i-b_i=0$ implies $dy_i=\frac{b_i'-a_i'y_i}{a_i}dt$. Hence we can write $\omega=fdt$ with some $f\in K(F_I)$. Moreover, since $dt\in\Omega^1_{F_I/\mathbb{K}}$ is nowhere vanishing, for any $c\in F_I$ we have

$$\mathrm{ord}_{x=c}(fdt)=\mathrm{ord}_{x=c}(f). \tag{10.3}$$

By Lemma 10.1, the ring $\mathcal{O}(F_I)$ is a PID whose prime element is t or $t-\frac{1}{\alpha-a}$ with some $\alpha\in F_I\backslash\{\infty\}$. Thus we can write

$$f=\frac{g}{\prod_{j=1}^s(t-\beta_j)^{n_j}}$$

with some $n_j\in\mathbb{Z}_{\geqslant0}$, $g\in\mathcal{O}(F_I)$ and $\beta_j\in\mathbb{K}$ satisfying

$$\alpha_j:=(\beta_j:-(1+a\beta_j))\in F_I,\quad t-\beta_j\nmid g.$$

When $\beta_j\neq0$, we have $\alpha_j\in\mathbb{K}$ and

$$t-\beta_j=\frac{1}{x-a}-\frac{1}{\alpha_j-a}=-\frac{x-\alpha_j}{(\alpha_j-a)(x-a)}.$$

For any $j=1,\ldots,s$ and $c\in F_I$, this implies

$$\mathrm{ord}_{x=c}(t-\beta_j)=\begin{cases}1 & (c=\alpha_j),\\ 0 & (c\neq\alpha_j).\end{cases} \tag{10.4}$$

Since ω has no poles on ∂D_i, (10.3) and (10.4) yield $\alpha_j\in F_I^\circ$ so that $t-\beta_j\in\mathcal{O}(\partial D_i)^\times$ for all $i=1,\ldots,r$.

By Lemma 10.23, there exists a sequence $\{g_j\}_{j\geqslant0}$ in the ring $R(F_I)$ which converges to g with respect to the Banach topology of $\mathcal{O}(F_I)$. By the continuity of Res_c as in Lemma 10.16 and that of the circular residue map $\mathrm{Res}_{\partial D_i} : \Omega^1_{\partial D_i/\mathbb{K}} \to \mathbb{K}$, we may assume that $g \in R(F_I)$ and thus ω is a meromorphic differential form on $\mathbb{P}^1_{\mathbb{K}}$ without poles on ∂D_i for all i.

Now the residue theorem on algebraic curves implies

$$\sum_{c \in \mathbb{P}^1_{\mathbb{K}}} \mathrm{Res}_c(\omega) = 0.$$

Since $D_1, \dots, D_r$ are disjoint, this yields

$$\begin{aligned} 0 = \sum_{c \in \mathbb{P}^1_{\mathbb{K}}} \mathrm{Res}_c(\omega) &= \sum_{c \in F_I^\circ} \mathrm{Res}_c(\omega) + \sum_{i=1}^{r} \sum_{c \in D_i} \mathrm{Res}_c(\omega) \\ &= \sum_{c \in F_I^\circ} \mathrm{Res}_c(\omega) + \sum_{i=1}^{r} \sum_{c \in D_i^\circ} \mathrm{Res}_c(\omega) \end{aligned}$$

and the theorem follows from Lemma 10.17. □

11. Harmonic cocycles and Drinfeld modular forms

11.1. Annular residue. Let $e \in \mathcal{T}_1^o$ and let $\mathcal{V}(e)$ be the annulus in $\mathbb{P}^1(\mathbb{C}_\infty)$ as in Definition 5.13. Recall that we have

$$\mathcal{V}(e) = \mathbb{P}^1(\mathbb{C}_\infty) \backslash (\mathcal{U}(e) \sqcup \mathcal{U}(-e)) = \mathcal{V}(-e).$$

Here $\mathcal{U}(e)$ is the distinguished closed disc in $\mathbb{P}^1(\mathbb{C}_\infty)$ associated with the edge e (Definition 4.19). Let $a \in K_\infty$ be a center of $U(e)$, so that for some $\rho \in q^{\mathbb{Z}}$ we have

$$\{\mathcal{U}(e), \mathcal{U}(-e)\} = \{D_{\mathbb{C}_\infty}(a, \rho), D'_{\mathbb{C}_\infty}(a, q\rho)\}.$$

Thus we can write

$$\mathcal{V}(e) = \{z \in \mathbb{C}_\infty \mid \rho < |z - a| < q\rho\}.$$

Definition 11.1. For any $\sigma \in q^{\mathbb{Q}}$ satisfying $\rho < \sigma < q\rho$, put

$$C_\sigma(e) = \{z \in \mathbb{C}_\infty \mid |z - a| = \sigma\}$$

and call it the concentric circle in $\mathcal{V}(e)$ of radius σ.

Lemma 11.2. *Let $a, b \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$. Suppose*

$$\{z \in \mathbb{C}_\infty \mid \rho < |z - a| < q\rho\} = \{z \in \mathbb{C}_\infty \mid \rho < |z - b| < q\rho\}.$$

Then for any $\sigma \in q^{\mathbb{Q}}$ satisfying $\rho < \sigma < q\rho$, we have

$$\{z \in \mathbb{C}_\infty \mid |z - a| = \sigma\} = \{z \in \mathbb{C}_\infty \mid |z - b| = \sigma\}.$$

In particular, the concentric circle $C_\sigma(e)$ is independent of the choice of a center a of $U(e)$.

Proof. First we claim $|a-b| \leqslant \rho$. Indeed, taking the complement we see that for any $z \in \mathbb{C}_\infty$,

$$(|z-a| \leqslant \rho \text{ or } |z-a| \geqslant q\rho) \quad \Leftrightarrow \quad (|z-b| \leqslant \rho \text{ or } |z-b| \geqslant q\rho).$$

For $z = a$ this yields $|a-b| \leqslant \rho$ or $|a-b| \geqslant q\rho$. Suppose $|a-b| \geqslant q\rho$. Take any $z \in \mathbb{C}_\infty$ satisfying $\rho < |z-a| < q\rho$. Then we have

$$|z-b| = |z-a+(a-b)| = |a-b| \geqslant q\rho,$$

which is a contradiction.

Now, for any $z \in \mathbb{C}_\infty$ satisfying $|z-a| = \sigma$, we have

$$|z-b| = |z-a+(a-b)| = |z-a| = \sigma$$

and the lemma follows. □

Let $C_\sigma(e)$ be the concentric circle of radius σ in $\mathcal{V}(e)$. Then

$$C_\sigma(e) = \partial_a D_{\mathbb{C}_\infty}(a,\sigma) = \partial_a D'_{\mathbb{C}_\infty}(a,\sigma).$$

By the inequality $\rho < \sigma < q\rho$, we have

$$\begin{aligned} D_{\mathbb{C}_\infty}(a,\rho) &\subseteq D_{\mathbb{C}_\infty}(a,\sigma) \not\supseteq D'_{\mathbb{C}_\infty}(a,q\rho), \\ D'_{\mathbb{C}_\infty}(a,q\rho) &\subseteq D'_{\mathbb{C}_\infty}(a,\sigma) \not\supseteq D_{\mathbb{C}_\infty}(a,\rho). \end{aligned}$$

Thus there exists a unique element $D_\sigma \in \{D_{\mathbb{C}_\infty}(a,\sigma), D'_{\mathbb{C}_\infty}(a,\sigma)\}$ satisfying $\mathcal{U}(e) \subseteq D_\sigma$.

Lemma 11.3. *The closed disc D_σ is independent of the choice of a center of $U(e)$.*

Proof. Let a, b be centers of $U(e)$. Note that $D'_{\mathbb{C}_\infty}(a,\rho) = D'_{\mathbb{C}_\infty}(b,\rho)$ implies $|a-b| < \rho$. Thus we have $|a-b| \leqslant \rho < \sigma$ and Lemma 4.2 shows $D_{\mathbb{C}_\infty}(a,\sigma) = D_{\mathbb{C}_\infty}(b,\sigma)$ and $D'_{\mathbb{C}_\infty}(a,\sigma) = D'_{\mathbb{C}_\infty}(b,\sigma)$. □

Definition 11.4. Let $C_\sigma(e)$ be the concentric circle of radius σ in $\mathcal{V}(e)$. Let $a \in K_\infty$ be a center of $U(e)$ and let $D_\sigma \in \{D_{\mathbb{C}_\infty}(a,\sigma), D'_{\mathbb{C}_\infty}(a,\sigma)\}$ be the unique element satisfying $\mathcal{U}(e) \subseteq D_\sigma$. For $D_\sigma^\circ := \mathrm{Int}_a D_\sigma$, we also have $\mathcal{U}(e) \subseteq D_\sigma^\circ$. We call D_σ the canonical closed disc of radius σ for the edge e. We refer to the orientation ψ_{D_σ} of $C_\sigma(e)$ as the canonical orientation of concentric circles in $\mathcal{V}(e)$ for the edge e.

Definition 11.5. Let $f \in \mathcal{O}(\mathcal{V}(e))$ and consider the holomorphic differential form $f dz$ on $\mathcal{V}(e)$. Take any concentric circle $C_\sigma(e)$ in $\mathcal{V}(e)$ and equip it with the canonical orientation ψ_{D_σ} for e. Then we define

$$\mathrm{Res}_e(f dz) := \mathrm{Res}_{(C_\sigma(e),\psi_{D_\sigma})}(f dz)$$

and call it the annular residue of $f dz$ for the edge e.

Lemma 11.6. $\mathrm{Res}_e(fdz)$ *is independent of the choice of a concentric circle* $C_\sigma(e)$ *in* $\mathcal{V}(e)$.

Proof. Let $a \in K_\infty$ be a center of $U(e)$. Take any $\sigma \neq \sigma' \in q^{\mathbb{Q}}$ satisfying $\sigma, \sigma' \in (\rho, q\rho)$. By exchanging them if necessary, we may assume

$$\mathcal{U}(e) \subseteq D_{\sigma'} \subseteq D_\sigma^\circ = \mathrm{Int}_a D_\sigma.$$

Let $D'_\sigma \in \{D_{\mathbb{C}_\infty}(a,\sigma), D'_{\mathbb{C}_\infty}(a,\sigma)\}\backslash\{D_\sigma\}$, so that the orientation ψ_{D_σ} of the circle $C_\sigma(e)$ is opposite to $\psi_{D'_\sigma}$. Then applying Theorem 10.25 to $I = \{D_{\sigma'}, D'_\sigma\}$ we obtain

$$0 = \mathrm{Res}_{\partial_a D_{\sigma'}}(fdz) + \mathrm{Res}_{\partial_a D'_\sigma}(fdz) = \mathrm{Res}_{C_{\sigma'}(e)}(fdz) - \mathrm{Res}_{C_\sigma(e)}(fdz).$$

This concludes the proof. $\square$

Lemma 11.7.

$$\mathrm{Res}_{-e}(fdz) = -\mathrm{Res}_e(fdz).$$

Proof. Take a concentric circle C_σ in $\mathcal{V}(e) = \mathcal{V}(-e)$. For a center $a \in K_\infty$ of $U(e)$, write

$$\{D_\sigma, D'_\sigma\} = \{D_{\mathbb{C}_\infty}(a,\sigma), D'_{\mathbb{C}_\infty}(a,\sigma)\} \quad \text{with} \quad \mathcal{U}(e) \subseteq D_\sigma.$$

Then we have $\mathcal{U}(-e) \subseteq D'_\sigma$. Thus the canonical orientation of C_σ for the edge $-e$ is the opposite of that for e. This yields

$$\mathrm{Res}_{-e}(fdz) = \mathrm{Res}_{(C_\sigma, \psi_{D'_\sigma})}(fdz) = -\mathrm{Res}_{(C_\sigma,\psi_{D_\sigma})}(fdz) = -\mathrm{Res}_e(fdz).$$

$\square$

Lemma 11.8. *For any* $\gamma \in GL_2(K)$, *let* $\gamma^*(fdz)$ *be the pull-back of* fdz *by the Möbius transformation* $\gamma : \Omega \to \Omega$. *Then we have*

$$\mathrm{Res}_{\gamma \circ e}(fdz) = \mathrm{Res}_e(\gamma^*(f(dz))).$$

Proof. Write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$$\{U(e), U(-e)\} = \{D(z,\rho), D'(z,q\rho)\}$$

with some $z \in K_\infty$ and $\rho \in q^{\mathbb{Z}}$. Take any $\sigma \in q^{\mathbb{Q}}$ satisfying $\rho < \sigma < q\rho$. Let

$$C_\sigma = \{x \in \mathbb{C}_\infty \mid |x - z| = \sigma\}$$

be the concentric circle of radius σ in $\mathcal{V}(e)$ and let $D_\sigma \in \{D_{\mathbb{C}_\infty}(z,\sigma), D'_{\mathbb{C}_\infty}(z,\sigma)\}$ be the canonical closed disc of radius σ for e. Then we have

$$\gamma(C_\sigma) \subseteq \gamma(\mathcal{V}(e)) = \mathcal{V}(\gamma\circ e), \quad \mathcal{U}(\gamma \circ e) = \gamma(\mathcal{U}(e)) \subseteq \gamma(D_\sigma).$$

First we claim that $\gamma(C_\sigma)$ is a concentric circle in $\mathcal{V}(\gamma\circ e)$ and $\gamma(D_\sigma)$ is a closed disc in $\mathbb{P}^1_{\mathbb{C}_\infty}$ satisfying $\gamma(C_\sigma)=\partial_w\gamma(D_\sigma)$ with some center w of $U(\gamma\circ e)$. Note that

$$\gamma(\mathcal{V}(e))=\mathbb{P}^1(\mathbb{C}_\infty)\backslash\left(\gamma(D_{\mathbb{C}_\infty}(z,\rho))\sqcup\gamma(D'_{\mathbb{C}_\infty}(z,q\rho))\right).$$

Since $z\in K_\infty$, we have either $|cz+d|\geqslant q\rho|c|$ or $|cz+d|\leqslant\rho|c|$. If $|cz+d|\geqslant q\rho|c|$, then by Lemma 4.7, Lemma 4.8 and Lemma 4.9 we have

$$\begin{aligned}\{\gamma(U(e)),\gamma(U(-e))\}&=\left\{D\left(\gamma(z),\rho\frac{|ad-bc|}{|cz+d|^2}\right),D'\left(\gamma(z),q\rho\frac{|ad-bc|}{|cz+d|^2}\right)\right\},\\ \gamma(\mathcal{V}(e))&=\left\{x\in\mathbb{C}_\infty\ \middle|\ \rho\frac{|ad-bc|}{|cz+d|^2}<|x-\gamma(z)|<q\rho\frac{|ad-bc|}{|cz+d|^2}\right\},\\ \gamma(C_\sigma)&=\left\{x\in\mathbb{C}_\infty\ \middle|\ |x-\gamma(z)|=\sigma\frac{|ad-bc|}{|cz+d|^2}\right\},\\ \gamma(D_\sigma)&\in\left\{D_{\mathbb{C}_\infty}\left(\gamma(z),\sigma\frac{|ad-bc|}{|cz+d|^2}\right),D'_{\mathbb{C}_\infty}\left(\gamma(z),\sigma\frac{|ad-bc|}{|cz+d|^2}\right)\right\}\end{aligned}$$

and γ induces an isomorphism

$$C_\sigma=\partial_zD_\sigma\to\partial_{\gamma(z)}\gamma(D_\sigma)=\gamma(C_\sigma).$$

Similarly, if $|cz+d|\leqslant\rho|c|$, then we have

$$\begin{aligned}\{\gamma(U(e)),\gamma(U(-e))\}&=\left\{D'\left(\frac{a}{c},\frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right),D\left(\frac{a}{c},\frac{1}{q\rho}\frac{|ad-bc|}{|c|^2}\right)\right\},\\ \gamma(\mathcal{V}(e))&=\left\{x\in\mathbb{C}_\infty\ \middle|\ \frac{1}{q\rho}\frac{|ad-bc|}{|c|^2}<\left|x-\frac{a}{c}\right|<\frac{1}{\rho}\frac{|ad-bc|}{|c|^2}\right\},\\ \gamma(C_\sigma)&=\left\{x\in\mathbb{C}_\infty\ \middle|\ \left|x-\frac{a}{c}\right|=\frac{1}{\sigma}\frac{|ad-bc|}{|c|^2}\right\},\\ \gamma(D_\sigma)&\in\left\{D'_{\mathbb{C}_\infty}\left(\frac{a}{c},\frac{1}{\sigma}\frac{|ad-bc|}{|c|^2}\right),D_{\mathbb{C}_\infty}\left(\frac{a}{c},\frac{1}{\sigma}\frac{|ad-bc|}{|c|^2}\right)\right\}\end{aligned}$$

and γ induces an isomorphism

$$C_\sigma=\partial_zD_\sigma\to\partial_{\frac{a}{c}}\gamma(D_\sigma)=\gamma(C_\sigma).$$

Hence the claim follows.

Now Lemma 10.13 yields

$$\mathrm{Res}_{(C_\sigma,\psi_{D_\sigma})}(\gamma^*(f dz))=\mathrm{Res}_{(\gamma(C_\sigma),\psi_{\gamma(D_\sigma)})}(f dz),$$

from which the lemma follows. □

11.2. **Harmonic cocycles attached to Drinfeld modular forms.** Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$ be an integer.

Definition 11.9. For any $f \in \mathcal{O}(\Omega)$, we define a map $\mathrm{Res}(f) : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ by

$$\mathrm{Res}(f)(e)(X^i Y^{k-2-i}) := \mathrm{Res}_e((-z)^{k-2-i} f(z)dz).$$

Note that $(-z)^{k-2-i}$ is obtained by plugging in $(X,Y) = (1,-z)$ into $X^i Y^{k-2-i}$.

Lemma 11.10. *For any $\gamma \in GL_2(K)$, $e \in \mathcal{T}_1^o$ and $f \in \mathcal{O}(\Omega)$, we have*

$$\mathrm{Res}(f)(\gamma \circ e) = \gamma \circ \mathrm{Res}(f|_k \gamma)(e).$$

Proof. Take any integer $i \in [0, k-2]$. By Lemma 11.8, we have

$$\begin{aligned}\mathrm{Res}(f)(\gamma \circ e)(X^i Y^{k-2-i}) &= \mathrm{Res}_{\gamma \circ e}((-z)^{k-2-i} f(z)dz)\\ &= \mathrm{Res}_e(\gamma^*((-z)^{k-2-i} f(z)dz)).\end{aligned}$$

Then the differential form inside Res_e equals

$$\begin{aligned}\gamma^*((-z)^{k-2-i} f(z)dz) &= \left(-\frac{az+b}{cz+d}\right)^{k-2-i} f(\gamma(z)) d\left(\frac{az+b}{cz+d}\right)\\ &= \left(-\frac{az+b}{cz+d}\right)^{k-2-i} f(\gamma(z)) \frac{ad-bc}{(cz+d)^2} dz\\ &= \frac{(ad-bc)^{k-1}}{(cz+d)^k} f(\gamma(z))\\ &\qquad \cdot (cz+d)^i (-(az+b))^{k-2-i} (ad-bc)^{2-k} dz\\ &= (f|_k \gamma)(z)(cz+d)^i(-(az+b))^{k-2-i}(ad-bc)^{2-k} dz.\end{aligned}$$

This and (9.1) yield

$$\begin{aligned}&\mathrm{Res}(f)(\gamma \circ e)(X^i Y^{k-2-i})\\ &\qquad = \mathrm{Res}(f|_k\gamma)(e)((ad-bc)^{2-k}(dX-cY)^i(-bX+aY)^{k-2-i})\\ &\qquad = (\gamma \circ \mathrm{Res}(f|_k \gamma)(e))(X^i Y^{k-2-i}),\end{aligned}$$

from which the lemma follows. $\square$

Proposition 11.11. *For any $f \in M_k(\Gamma)$, we have*

$$\mathrm{Res}(f) \in C_k^{\mathrm{har}}(\Gamma).$$

Proof. By Lemma 11.7, we have $\mathrm{Res}(f)(-e) = -\mathrm{Res}(f)(e)$.

Next we prove the harmonicity, namely

$$\sum_{o(e)=v} \mathrm{Res}(f)(e) = 0$$

for any $v \in \mathcal{T}_0$. For this, put $\Lambda_v = \{e \in \mathcal{T}_1^o \mid o(e) = v\}$. Let $e \neq e' \in \Lambda_v$, so that $U(e) \cap U(e') = \varnothing$. By Lemma 4.6, we have

$$\mathcal{U}(e) \cap \mathcal{U}(e') = \varnothing. \tag{11.1}$$

Since $t(-e) = v = o(e')$, the equality (4.5) and Lemma 4.5 yield $\mathcal{U}(e') \subseteq \mathcal{U}(-e)$ and thus

$$\mathcal{V}(e) \cap \mathcal{U}(e') = \varnothing. \tag{11.2}$$

Since $d(t(e), t(e')) = 2$, Definition 5.13 and Lemma 5.15 imply

$$\mathcal{V}(e) \cap \mathcal{V}(e') = \mathcal{U}(v) \cap \mathcal{U}(t(e)) \cap \mathcal{U}(t(e')) = \varnothing. \tag{11.3}$$

For any $e \in \Lambda_v$, take a concentric circle $C(e)$ of some radius σ in $\mathcal{V}(e)$. Let $D(e)$ be the canonical closed disc of radius σ for e. Then

$$\mathcal{U}(e) \subseteq D(e)^\circ, \quad D(e) \subseteq \mathcal{V}(e) \cup \mathcal{U}(e).$$

By (11.1), (11.2) and (11.3), this implies that $D(e) \cap D(e') = \varnothing$ for any $e \neq e' \in \Lambda_v$.

We equip the circle $C(e)$ with the canonical orientation for e, namely the orientation $\psi_{D(e)}$ associated with $D(e)$. Put $I = \{D(e) \mid e \in \Lambda_v\}$. Let $D(e)^\circ$ be the interior of $D(e)$ for a center of $U(e)$ and consider the connected affinoid variety

$$F_I = \mathbb{P}^1(\mathbb{C}_\infty) \backslash \coprod_{o(e)=v} D(e)^\circ.$$

Then Definition 5.1 and Lemma 5.4 imply $F_I \subseteq \mathcal{U}(v) \subseteq \Omega$ and thus the differential form $\omega = (-z)^{k-2-i} f(z) dz$ is holomorphic on F_I. Now Theorem 10.25 yields

$$\sum_{o(e)=v} \mathrm{Res}_e(\omega) = \sum_{o(e)=v} \mathrm{Res}_{\partial D(e)}(\omega) = 0$$

and the harmonicity follows.

Finally, since $f \in M_k(\Gamma)$, we have $f|_k \gamma = f$ for any $\gamma \in \Gamma$ and Lemma 11.10 yields

$$\mathrm{Res}(f)(\gamma \circ e) = \gamma \circ \mathrm{Res}(f|_k \gamma) = \gamma \circ \mathrm{Res}(f)$$

and $\mathrm{Res}(f)$ is Γ-equivariant. This concludes the proof. □

11.3. **Drinfeld cusp forms associated with harmonic cocycles.** Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$ be an integer. Let $\mathscr{A}_k$ be the set of locally meromorphic functions on $\mathbb{P}^1(K_\infty)$ of Definition 9.14.

Lemma 11.12. *For any $z \in \Omega$, the function $x \mapsto \frac{1}{z-x}$ is an element of $\mathscr{A}_k$.*

Proof. Since $z \in \Omega$, the function $\frac{1}{z-x}$ has no pole on K_∞. Take any $x \in K_\infty$. Then $z - x \neq 0$ and $|z - x| > 0$. Around x we have

$$\frac{1}{z-X} = \frac{1}{z-x-(X-x)} = \frac{1}{z-x} \cdot \frac{1}{1-\frac{X-x}{z-x}}$$
$$= \frac{1}{z-x} \sum_{n \geqslant 0} \left(\frac{X-x}{z-x} \right)^n,$$

which is analytic on $D_{\mathbb{C}_\infty}(x, q^{-n})$ for any $n \in \mathbb{Z}$ satisfying $q^{-n} < |z - x|$.

Around $x = \infty$, we have

$$\frac{1}{z-X} = \frac{1}{X} \cdot \frac{-1}{1-\frac{z}{X}} = \frac{-1}{X} \sum_{n \geqslant 0} \left(\frac{z}{X} \right)^n,$$

which is analytic on $D_{\mathbb{C}_\infty}(\infty, q^{-n}) = D'_{\mathbb{C}_\infty}(0, q^n)$ for any $n \in \mathbb{Z}$ satisfying $q^n > |z|$.

□

Definition 11.13. For any $c \in C_k^{\mathrm{har}}(\Gamma)$, using the integration of Theorem 9.16 we define a function $F_c : \Omega \to \mathbb{C}_\infty$ by

$$F_c(z) := \int_{\mathbb{P}^1(K_\infty)} \frac{1}{z-x} d\mu_c(x).$$

Lemma 11.14. *We have $F_c \in \mathcal{O}(\Omega)$. Moreover, there exists a constant C_1 such that for any sufficiently small integer r we have*

$$\sup_{z \in \Omega_r} |F_c(z)| \leqslant C_1 q^{\frac{k}{2} r}.$$

Proof. For any integers r, s satisfying $s \geqslant -r$, consider the affinoid variety $\Omega_{r,s}$ of Definition 5.19. By Lemma 5.20, we can write

$$\mathbb{P}^1(K_\infty) = D^\circ(\infty, q^{-s}) \sqcup \coprod_{a \in J} D^\circ(a, q^{-r}) = D(\infty, q^{-s-1}) \sqcup \coprod_{a \in J} D(a, q^{-r-1}),$$
$$\Omega_{r,s} = \mathbb{P}^1(\mathbb{C}_\infty) \backslash \left(D^\circ_{\mathbb{C}_\infty}(\infty, q^{-s}) \sqcup \coprod_{a \in J} D^\circ_{\mathbb{C}_\infty}(a, q^{-r}) \right)$$

for some finite subset $J \subseteq K_\infty$. Put

$$F_{s,\infty}(z) = \int_{D(\infty, q^{-s-1})} \frac{1}{z-x} d\mu_c(x), \quad F_{r,a}(z) = \int_{D(a, q^{-r-1})} \frac{1}{z-x} d\mu_c(x).$$

By Theorem 9.16 (1), we have

$$F_c(z) = F_{s,\infty}(z) + \sum_{a \in J} F_{r,a}(z).$$

First consider $F_{s,\infty}$. For any $z \in \Omega_{r,s}$, we have $|z| \leqslant q^s$ and the function

$$\frac{1}{z-x} = \frac{-1}{x} \cdot \frac{1}{1-\frac{z}{x}} = -\sum_{n\geqslant 0} \frac{z^n}{x^{n+1}}$$

lies in $\mathbb{C}_\infty\langle \frac{1}{\pi_\infty^{s+1} x}\rangle$. Then Theorem 9.16 (4) implies

$$F_{s,\infty}(z) = -\sum_{n\geqslant 0} \left(\int_{D(\infty,q^{-s-1})} \frac{1}{x^{n+1}} d\mu_c(x) \right) z^n \quad \text{if} \quad |z| \leqslant q^s.$$

In particular, for $a_n = \int_{D(\infty,q^{-s-1})} \frac{1}{x^{n+1}} d\mu_c(x)$ the power series $-\sum_{n\geqslant 0} a_n z^n$ converges on $D_{\mathbb{C}_\infty}(0,q^s)$. Thus $F_{s,\infty}$ is the restriction of an element $\mathcal{O}(D_{\mathbb{C}_\infty}(0,q^s))$ to $\Omega_{r,s} \subseteq D_{\mathbb{C}_\infty}(0,q^s)$, which gives $F_{s,\infty} \in \mathcal{O}(\Omega_{r,s})$. By Theorem 9.16 (3), with the constant C of the theorem we obtain

$$\sup_{z\in\Omega_{r,s}} |F_{s,\infty}(z)| \leqslant C \sup_{n\geqslant 0} q^{(-s-1)(n+1+\frac{k-2}{2})+ns} = Cq^{-\frac{k}{2}(s+1)}.$$

Next consider $F_{r,a}$. For any $z \in \Omega_{r,s}$, we have $|z-a| \geqslant q^{-r}$ and the function

$$\frac{1}{z-x} = \frac{1}{z-a} \cdot \frac{1}{1-\frac{x-a}{z-a}} = \sum_{n\geqslant 0} \frac{(x-a)^n}{(z-a)^{n+1}}$$

lies in $\mathbb{C}_\infty\langle \frac{x-a}{\pi_\infty^{r+1}}\rangle$. Then Theorem 9.16 (4) implies

$$F_{r,a}(z) = \sum_{n\geqslant 0} \left(\int_{D(a,q^{-r-1})} (x-a)^n d\mu_c(x) \right) \frac{1}{(z-a)^{n+1}} \quad \text{if} \quad |z-a| \geqslant q^{-r}.$$

In particular, $F_{r,a}$ is the restriction of an element $\mathcal{O}(D'_{\mathbb{C}_\infty}(a,q^{-r}))$ to $\Omega_{r,s} \subseteq D'_{\mathbb{C}_\infty}(a,q^{-r})$, which gives $F_{r,a} \in \mathcal{O}(\Omega_{r,s})$. By Theorem 9.16 (3), we obtain

$$\sup_{z\in\Omega_{r,s}} |F_{r,a}(z)| \leqslant C \sup_{n\geqslant 0} q^{(-r-1)(n-\frac{k-2}{2})+(n+1)r} = Cq^{\frac{k}{2}(r+1)-1}.$$

Hence, we have $F_c \in \mathcal{O}(\Omega_{r,s})$ for any integers r, s satisfying $s \geqslant -r$. Since it holds for any integers $r, s \geqslant 0$, Proposition 5.24 yields $F_c \in \mathcal{O}(\Omega)$.

Moreover, we have

$$\sup_{z\in\Omega_{r,s}} |F_c(z)| \leqslant C \max\{q^{-\frac{k}{2}(s+1)}, q^{\frac{k}{2}(r+1)-1}\}.$$

Since $s \geqslant -r$ is arbitrary and $k > 0$, this yields

$$\sup_{z\in\Omega_r} |F_c(z)| \leqslant C \max\{q^{-\frac{k}{2}(-r+1)}, q^{\frac{k}{2}(r+1)-1}\} \leqslant Cq^{\frac{k-2}{2}} q^{\frac{k}{2}r},$$

which concludes the proof. $\square$

Lemma 11.15. *For any $\gamma \in GL_2(K)$, we have $F_{\gamma c}|_k\gamma = F_c$. In particular,*

$$F_c|_k\gamma = F_c \quad \text{for any } \gamma \in \Gamma.$$

Proof. Write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. First we claim that

$$\int_{\mathbb{P}^1(K_\infty)} \frac{(cx+d)^{k-2}}{\gamma(z)-\gamma(x)} d\mu_c(x) = \frac{(cz+d)^k}{ad-bc} F_c(z)$$

for any $z \in \Omega$.

Indeed, note that we have

$$\begin{aligned}\frac{(cz+d)^{k-2}}{\gamma(z)-\gamma(x)} &= \frac{(cx+d)^{k-1}(cz+d)}{(az+b)(cx+d)-(ax+b)(cz+d)} \\ &= \frac{(cx+d)^{k-1}(cz+d)}{(ad-bc)(z-x)}.\end{aligned}$$

This shows

$$\begin{aligned}&\frac{(cz+d)^{k-2}}{\gamma(z)-\gamma(x)} - \frac{(cz+d)^k}{(ad-bc)(z-x)} \\ &\qquad = \frac{(cz+d)}{ad-bc} \cdot \frac{(cx+d)^{k-1}-(cz+d)^{k-1}}{z-x}.\end{aligned}$$

Since we can write

$$(cx+d)^{k-1}-(cz+d)^{k-1} = (z-x)P(x)$$

with some $P(x) \in \mathbb{C}_\infty[x]$ satisfying $\deg(P) \leqslant k-2$, the claim follows from Lemma 9.7 (2).

Take any $v \in \mathcal{T}_0$. By the claim and Proposition 9.55, we have

$$\begin{aligned}F_{\gamma c}(\gamma(z)) &= \int_{\mathbb{P}^1(K_\infty)} \frac{1}{\gamma(z)-x} d\mu_{\gamma c}(x) \\ &= \sum_{o(e)=v} \int_{U(\gamma\circ e)} \frac{1}{\gamma(z)-x} d\mu_{\gamma c}(x) \\ &= \sum_{o(e)=v} \int_{U(e)} \frac{\det(\gamma)^{2-k}(cx+d)^{k-2}}{\gamma(z)-\gamma(x)} d\mu_c(x) \\ &= \int_{\mathbb{P}^1(K_\infty)} \frac{\det(\gamma)^{2-k}(cx+d)^{k-2}}{\gamma(z)-\gamma(x)} d\mu_c(x) \\ &= \det(\gamma)^{1-k}(cz+d)^k F_c(z).\end{aligned}$$

This concludes the proof. $\square$

Lemma 11.16. *For any arithmetic subgroup Γ of $GL_2(K)$, an integer $k \geqslant 2$ and $c \in C_k^{\mathrm{har}}(\Gamma)$, the rigid analytic function F_c vanishes at $[\infty]$.*

Proof. Since Lemma 11.14 yields

$$\lim_{r \to -\infty} \sup_{z \in \Omega_r} |F_c(z)| = 0,$$

Lemma 11.15 and Lemma 6.60 (2) conclude the proof. □

Lemma 11.17. *For any arithmetic subgroup Γ of $GL_2(K)$, an integer $k \geqslant 2$ and $c \in C_k^{\mathrm{har}}(\Gamma)$, we have*

$$F_c \in S_k(\Gamma).$$

Proof. Let $\nu = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K)$ and $s = \nu(\infty)$. Note that we have $c = {}^{\nu\nu^{-1}}c = {}^{\nu}({}^{\nu^{-1}}c)$.

By Lemma 11.15, we have

$$F_c|_k\nu = F_{{}^{\nu}({}^{\nu^{-1}}c)}|_k\nu = F_{{}^{\nu^{-1}}c}.$$

Since ${}^{\nu^{-1}}c \in C_k^{\mathrm{har}}(\nu^{-1}\Gamma\nu)$, Lemma 11.16 and Lemma 6.60 imply that $F_c|_k\nu = F_{{}^{\nu^{-1}}c}$ vanishes at $[\infty]$, namely F_c vanishes at $[s]$. □

Lemma 11.18. *Let $a \in K_\infty$ and $\rho, \eta \in |K_\infty^\times|$. Suppose $\eta < \min\{\rho, \rho^{-1}\}$ and $|a| < \eta^{-1}$. Then there exists a finite subset $\Lambda \subseteq K_\infty$ satisfying*

$$D'(a,\rho) = D(\infty,\eta) \sqcup \coprod_{a' \in \Lambda} D(a',\eta).$$

Proof. Since $\min\{\rho,\rho^{-1}\} \leqslant 1$, we have $\eta < 1$ and $\eta < \eta^{-1}$.

First we show $D(\infty,\eta) \subseteq D'(a,\rho)$. Let $x \in K_\infty$ satisfy $|x| \geqslant \eta^{-1}$. Then $|x-a| = |x| \geqslant \eta^{-1} > \rho$ and $x \in D'(a,\rho)$.

Next we show $D(a',\eta) \subseteq D'(a,\rho) \backslash D(\infty,\eta)$ for any $a' \in K_\infty$ satisfying $|a'-a| \geqslant \rho$ and $|a'| < \eta^{-1}$. Let $x \in K_\infty$ satisfy $|x-a'| \leqslant \eta$. Then

$$|x-a| = |x-a'+(a'-a)| = |a'-a| \geqslant \rho$$

and $x \in D'(a,\rho)$. Moreover, we have $|x| = |x-a'+a'| < \eta^{-1}$ and $x \notin D(\infty,\eta)$. Now Lemma 4.2 (1) concludes the proof. □

Lemma 11.19. *Let $c \in C_k^{\mathrm{har}}(\Gamma)$ and let $e \in \mathcal{T}_1^o$. Suppose $\infty \notin U(e)$. Write $U(e) = D(a,\rho)$ with some $a \in K_\infty$ and $\rho \in |K_\infty^\times|$. Then for any integer $m \geqslant 0$, we have*

$$\mathrm{Res}_e((z-a)^m F_c(z)dz) = \int_{U(e)} (x-a)^m d\mu_c(x).$$

Proof. For any $\sigma \in q^{\mathbb{Q}}$, choose $\varpi_\sigma \in \mathbb{C}_\infty$ satisfying $|\varpi_\sigma| = \sigma$. Write

$$F_c(z) = \int_{U(e)} \frac{1}{z-x} d\mu_c(x) + \int_{U(-e)} \frac{1}{z-x} d\mu_c(x).$$

To compute Res_e, take any $\sigma \in q^{\mathbb{Q}} \cap (\rho, q\rho)$ and consider the concentric circle of radius σ in $\mathcal{V}(e)$

$$C_\sigma = \{z \in \mathbb{C}_\infty \mid |z-a| = \sigma\}.$$

Then its canonical orientation for the edge e is given by its parameter $\frac{z-a}{\varpi_\sigma}$.

For any $z \in \mathcal{V}(e)$, we have $\rho < |z-a| < q\rho$ and

$$\frac{1}{z-x} = \frac{1}{z-a} \cdot \frac{1}{1 - \frac{x-a}{z-a}} = \sum_{n \geqslant 0} \frac{(x-a)^n}{(z-a)^{n+1}} \in \mathbb{C}_\infty \left\langle \frac{x-a}{\varpi_\rho} \right\rangle$$

and Theorem 9.16 (4) yields

$$\int_{U(e)} \frac{1}{z-x} d\mu_c(x) = \sum_{n \geqslant 0} \frac{1}{(z-a)^{n+1}} \int_{U(e)} (x-a)^n d\mu_c(x). \tag{11.4}$$

Since the series converges for any $z \in C_\sigma$, the right-hand side of (11.4) lies in $\mathcal{O}(C_\sigma)$ and

$$\mathrm{Res}_{C_\sigma} \left((z-a)^m \int_{U(e)} \frac{1}{z-x} d\mu_c(x) dz \right) = \int_{U(e)} (x-a)^m d\mu_c(x).$$

Now it is enough to show

$$\mathrm{Res}_{C_\sigma} \left((z-a)^m \int_{U(-e)} \frac{1}{z-x} d\mu_c(x) dz \right) = 0.$$

For this, choose $\eta \in q^{\mathbb{Z}}$ satisfying $\eta < \min\{q\rho, (q\rho)^{-1}\}$ and $|a| < \eta^{-1}$. By Lemma 11.18, we can find a decomposition

$$U(-e) = D'(a, q\rho) = D(\infty, \eta) \sqcup \coprod_{a' \in \Lambda} D(a', \eta)$$

with some finite subset $\Lambda \subseteq K_\infty$. Then we have

$$\int_{U(-e)} \frac{1}{z-x} d\mu_c(x) = \int_{D(\infty,\eta)} \frac{1}{z-x} d\mu_c(x) + \sum_{a' \in \Lambda} \int_{D(a',\eta)} \frac{1}{z-x} d\mu_c(x).$$

Note that a' satisfies $\sigma < q\rho \leqslant |a'-a| \leqslant (q\eta)^{-1} < \eta^{-1}$.

For any $z \in \mathcal{V}(e)$, we have $\rho < |z-a| < q\rho$ and

$$|z-a'| = |z-a+(a-a')| = |a-a'| \geqslant q\rho.$$

Since $\eta < q\rho$, we have

$$\frac{1}{z-x} = \frac{1}{z-a'} \cdot \frac{1}{1-\frac{x-a'}{z-a'}} = \sum_{n\geqslant 0} \frac{(x-a')^n}{(z-a')^{n+1}} \in \mathbb{C}_\infty \left\langle \frac{x-a'}{\varpi_\eta} \right\rangle$$

and Theorem 9.16 (4) yields

$$\int_{D(a',\eta)} \frac{1}{z-x} d\mu_c(x) = \sum_{n\geqslant 0} \frac{1}{(z-a')^{n+1}} \int_{D(a',\eta)} (x-a')^n d\mu_c(x). \tag{11.5}$$

We claim that the right-hand side of (11.5) lies in $\mathbb{C}_\infty\langle\frac{z-a}{\varpi_\sigma}\rangle$. Indeed, consider the supremum norm on this affinoid algebra, which is a valuation by [BGR, Corollary 5.1.4/6]. Then $|z-a|_{\sup} = \sigma$ and

$$|z-a'|_{\sup} = |z-a+(a-a')|_{\sup} = |a'-a| \geqslant q\rho.$$

Moreover, since $|a'-a| > \sigma$ we have $z-a' \in \mathbb{C}_\infty\langle\frac{z-a}{\varpi_\sigma}\rangle^\times$. Now Theorem 9.16 (3) implies

$$\left|(z-a')^{-n} \int_{D(a',\eta)} (x-a')^n d\mu_c(x)\right|_{\sup} \leqslant (q\rho)^{-n} C\eta^{n-\frac{k-2}{2}} = C\eta^{-\frac{k-2}{2}} \left(\frac{\eta}{q\rho}\right)^n,$$

which goes to zero when $n \to \infty$ and the claim follows. Since $m \geqslant 0$, we obtain

$$\mathrm{Res}_{C_\sigma}\left((z-a)^m \int_{D(a',\eta)} \frac{1}{z-x} d\mu_c(x) dz\right) = 0.$$

Finally, let us consider the integration over $D(\infty,\eta)$. Since $|a| < \eta^{-1}$, for any $z \in \mathcal{V}(e)$ we have $\rho < |z-a| < q\rho < \eta^{-1}$ and

$$|z| = |z-a+a| < \eta^{-1}.$$

This implies

$$\frac{1}{z-x} = \frac{-1}{x} \cdot \frac{1}{1-\frac{z}{x}} = -\sum_{n\geqslant 0} \frac{z^n}{x^{n+1}} \in \mathbb{C}_\infty \left\langle \frac{1}{\varpi_\eta x} \right\rangle$$

and Theorem 9.16 (4) yields

$$\int_{D(\infty,\eta)} \frac{1}{z-x} d\mu_c(x) = -\sum_{n\geqslant 0} z^n \int_{D(\infty,\eta)} \frac{1}{x^{n+1}} d\mu_c(x). \tag{11.6}$$

We claim that the right-hand side of (11.6) lies in $\mathbb{C}_\infty\langle\frac{z-a}{\varpi_\sigma}\rangle$. Indeed, for the supremum norm of this affinoid algebra, we have

$$|z|_{\sup} = |z-a+a|_{\sup} < \eta^{-1}$$

and Theorem 9.16 (3) yields

$$\left|z^n \int_{D(\infty,\eta)} \frac{1}{x^{n+1}} d\mu_c(x)\right|_{\sup} = |z|_{\sup}^n C\eta^{n+1+\frac{k-2}{2}} = C\eta^{\frac{k}{2}} (|z|_{\sup}\eta)^n \to 0$$

when $n \to \infty$ and the claim follows. Since $m \geqslant 0$, we obtain

$$\mathrm{Res}_{C_\sigma}\left((z-a)^m \int_{D(\infty,\eta)} \frac{1}{z-x} d\mu_c(x) dz\right) = 0.$$

This concludes the proof. ☐

Proposition 11.20. *For any $c \in C_k^{\mathrm{har}}(\Gamma)$, we have $\mathrm{Res}(F_c) = c$. In particular, the map*

$$C_k^{\mathrm{har}}(\Gamma) \to S_k(\Gamma), \quad c \mapsto F_c$$

is an injective $\mathbb{C}_\infty$-linear map.

Proof. Take any $e \in \mathcal{T}_1^o$. We need to show $\mathrm{Res}(F_c)(e) = c(e)$. Replacing e by $-e$ if necessary, we may assume $\infty \notin U(e)$. Let i be any integer satisfying $0 \leqslant i \leqslant k-2$. Write $U(e) = D(a, \rho)$ with some $a \in K_\infty$ and $\rho \in |K^\times|$. Put

$$(-z)^{k-2-i} = \sum_{m=0}^{k-2} c_m (z-a)^m, \quad c_m \in K_\infty.$$

By Definition 11.9 and Lemma 11.19, we have

$$\begin{aligned}
\mathrm{Res}(F_c)(e)(X^i Y^{k-2-i}) &= \mathrm{Res}_e((-z)^{k-2-i} F_c(z) dz) \\
&= \sum_{m=0}^{k-2} c_m \mathrm{Res}_e((z-a)^m F_c(z) dz) \\
&= \sum_{m=0}^{k-2} c_m \int_{U(e)} (x-a)^m d\mu_c(x) \\
&= \int_{U(e)} (-x)^{k-2-i} d\mu_c(x).
\end{aligned}$$

By (9.2), this equals $c(e)(X^i Y^{k-2-i})$ and we obtain $\mathrm{Res}(F_c) = c$. Hence the map of the proposition is injective. Its $\mathbb{C}_\infty$-linearity follows from Corollary 9.17. ☐

12. Description of Drinfeld cuspforms via harmonic cocycles and the Steinberg module

Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$. In this section, we show that the injection $C_k^{\mathrm{har}}(\Gamma) \to S_k(\Gamma)$ of Proposition 11.20 for Γ is an isomorphism. Moreover, under the assumption that Γ is p'-torsion free, we give a description of $C_k^{\mathrm{har}}(\Gamma)$, and thus of $S_k(\Gamma)$, using the Steinberg module St.

For this, first we recall the notion of Γ-stable simplices and their properties, following [Ser, Ch. II, §2.9].

12.1. **Stable and unstable simplices.** In this subsection, we assume that Γ is p'-torsion free.

Definition 12.1. We say a vertex (*resp.* an edge) s of $\mathcal{T}$ is Γ-stable if $\mathrm{Stab}_\Gamma(s) = \{\mathrm{id}\}$, and Γ-unstable if not. We denote by $\mathcal{T}_0^{\Gamma\text{-st}}$ (*resp.* $\mathcal{T}_1^{o,\Gamma\text{-st}}$) the set of Γ-stable vertices (*resp.* edges), and by $\mathcal{T}_0^{\Gamma\text{-un}}$ (*resp.* $\mathcal{T}_1^{o,\Gamma\text{-un}}$) the set of Γ-unstable vertices (*resp.* edges).

Note that (3.2) implies that for any $g \in GL_2(K)$, a simplex s is Γ-stable if and only if $g \circ s$ is $g\Gamma g^{-1}$-stable. On the other hand, (3.1) implies that if $e \in \mathcal{T}_1^o$ is Γ-unstable, then $-e$, $o(e)$ and $t(e)$ are all Γ-unstable. Thus Γ-unstable vertices and edges form a subgraph of the tree $\mathcal{T}$ which we denote by $\mathcal{T}_\infty$.

Lemma 12.2. *Let $G \subseteq GL_2(K)$ be a nontrivial finite p-subgroup. Then there exists a unique rational end of $\mathcal{T}$ that is fixed by G.*

Proof. Note that the set of rational ends $\mathrm{End}_K(\mathcal{T})$ is identified with $\mathbb{P}^1(K)$ via the isomorphism of Lemma 2.10. Thus it is enough to show that there exists a unique line in K^2 that is fixed by G.

Since G is a nontrivial finite p-group, its center Z is nontrivial. Since Z is a nontrivial finite abelian p-group, we can find $g \in Z$ of order p. Since $g \in GL_2(K)$ satisfies $g^p = \mathrm{id}$, its minimal polynomial divides $(X-1)^p$ in the polynomial ring $K[X]$. Thus g has the eigenvalue one, and there exists a line $D \subseteq K^2$ which is fixed by g. Since $g \neq \mathrm{id}$, such a line is unique.

For any $h \in G$, we have $gh(D) = hg(D) = h(D)$ and the uniqueness of the line D yields $h(D) = D$. Thus D is stable under the action of G, and the action defines a character $\chi : G \to K^\times$. Since $\mathrm{char}(K) = p$, there is no nontrivial p-power roots of unity in K. Hence $\chi = 1$ and D is fixed by G. Since $G \neq \{\mathrm{id}\}$, such D is unique. This concludes the proof. $\square$

Lemma 12.3. *Suppose that Γ is an arithmetic subgroup of $GL_2(K)$ which is p'-torsion free. For any Γ-unstable simplex s of $\mathcal{T}$, the stabilizer subgroup $\mathrm{Stab}_\Gamma(s)$ is a nontrivial finite p-group.*

Proof. By Lemma 3.6, the group $\mathrm{Stab}_\Gamma(s)$ is finite. The assumption that Γ is p'-torsion free implies that it is a p-group. Since s is Γ-unstable, it is nontrivial. $\square$

Lemma 12.4. *Let $v \in \mathcal{T}_0^{\Gamma\text{-un}}$.*

(1) There exists a unique rational end $b(v) \in \mathrm{End}_K(\mathcal{T})$ fixed by $\mathrm{Stab}_\Gamma(v)$.
(2) For any $\gamma \in \Gamma$, we have $b(\gamma \circ v) = \gamma \circ b(v)$.

(3) *There exists a unique half-line $H(v)$ starting from v that represents $b(v)$.*
(4) *The half-line $H(v)$ is fixed by* $\mathrm{Stab}_\Gamma(v)$. *In particular, if we write $H(v) = \{w_i\}_{i \geqslant 0}$, then the edge $(w_i \to w_{i+1})$ is Γ-unstable for any i.*

Proof. By Lemma 12.3, the group $G := \mathrm{Stab}_\Gamma(v)$ is a nontrivial finite p-group. Then Lemma 12.2 yields (1).

For (2), by (3.2) we have $\mathrm{Stab}_\Gamma(\gamma \circ v) = \gamma \mathrm{Stab}_\Gamma(v)\gamma^{-1}$ for any $\gamma \in \Gamma$ and $\gamma \circ b(v)$ is a rational end fixed by this group. Hence the uniqueness in (1) yields $b(\gamma \circ v) = \gamma \circ b(v)$.

Since $\mathcal{T}$ is connected, we can find a half-line $H = \{w_i\}_{i \geqslant 0}$ starting with $w_0 = v$ and representing the end $b(v)$. If $H' = \{w'_i\}_{i \geqslant 0}$ is another such half-line, then H' agrees with H except finitely many vertices. Thus they yield a circuit unless $H = H'$. Since $\mathcal{T}$ is a tree, this shows $H = H'$ and (3) follows.

For any $g \in G$, put $g \circ H = \{g \circ w_i\}_{i \geqslant 0}$. Since G fixes v and $b(v)$, the uniqueness yields $H = g \circ H$. This implies $(g \circ w_i \to g \circ w_{i+1}) = (w_i \to w_{i+1})$ for any i and (4) also follows. □

Lemma 12.5. *Let $e \in \mathcal{T}_0^{\Gamma\text{-un}}$.*

(1) *There exists a unique rational end $b(e) \in \mathrm{End}_K(\mathcal{T})$ fixed by* $\mathrm{Stab}_\Gamma(e)$.
(2) *For any $w \in \{o(e), t(e)\}$, we have $b(e) = b(w)$.*

Proof. Since $\mathrm{Stab}_\Gamma(e)$ is a nontrivial finite p-group, Lemma 12.2 yields a unique rational end $b(e)$ that is fixed by $\mathrm{Stab}_\Gamma(e)$ and (1) follows.

For (2), note that w is Γ-unstable. Since we have

$$\mathrm{Stab}_\Gamma(e) \subseteq \mathrm{Stab}_\Gamma(w),$$

the rational end $b(w)$ of Lemma 12.4 (1) is fixed by $\mathrm{Stab}_\Gamma(e)$. Then the uniqueness of $b(e)$ implies $b(e) = b(w)$ and (2) follows. □

Lemma 12.6. *Let $b \in \mathrm{End}_K(\mathcal{T})$ be a rational end and let $H = \{w_n\}_{n \geqslant 0}$ be a half-line representing b. Then there exists an integer $N \geqslant 0$ such that for any $n \geqslant N$, the edge $f_n = (w_n \to w_{n+1})$ is Γ-unstable and* $\mathrm{Stab}_\Gamma(f_n) = \mathrm{Stab}_\Gamma(w_n) \subseteq \mathrm{Stab}_\Gamma(f_{n+1})$.

Proof. Take $\nu \in GL_2(K)$ satisfying $\nu\Gamma\nu^{-1} \subseteq GL_2(A)$. By (3.2), replacing H by $\nu \circ H$ we may assume that Γ is a congruence subgroup. By Lemma 3.15, there exists $\gamma \in \Gamma$ and $g \in GL_2(A)$ such that $\{\gamma g \circ v_n\}_{n \geqslant 0}$ and H agree up to finitely many vertices. Replacing Γ by the congruence subgroup $\gamma g \Gamma (\gamma g)^{-1}$, we may assume $H = \{v_n\}_{n \geqslant 0}$.

Take any $P \in A\backslash\mathbb{F}_q$ satisfying $\Gamma((P)) \subseteq \Gamma$. By Lemma 3.10 (2), for any $n \geqslant \deg(P)$ we have

$$\left\{\begin{pmatrix} 1 & \mathbb{F}_q P \\ 0 & 1 \end{pmatrix}\right\} \subseteq \mathrm{Stab}_\Gamma(v_n) = \mathrm{Stab}_\Gamma(e_n) \subseteq \mathrm{Stab}_\Gamma(v_{n+1}) = \mathrm{Stab}_\Gamma(e_{n+1}),$$

from which the lemma follows. $\square$

Lemma 12.7. *Let $\pi_0(\mathcal{T}_\infty)$ be the set of connected components of $\mathcal{T}_\infty$. For any $v \in \mathcal{T}_0^{\Gamma\text{-un}}$, let $[v]$ be the connected component of $\mathcal{T}_\infty$ that contains v. Then we have a Γ-equivariant bijection*

$$\pi_0(\mathcal{T}_\infty) \to \mathrm{End}_K(\mathcal{T}) \simeq \mathbb{P}^1(K), \quad [v] \mapsto b(v).$$

Proof. First we show that the map of the lemma is well-defined. Suppose that Γ-unstable vertices $v, v' \in \mathcal{T}_0$ satisfy $[v] = [v']$. This means that there exist vertices $w_0, \ldots, w_n$ of $\mathcal{T}_\infty$ satisfying $w_0 = v$, $w_n = v'$ and such that for any i the edge $(w_i \to w_{i+1})$ is Γ-unstable. Thus we may assume $n = 1$, so that $e = (v \to v')$ is a Γ-unstable edge. Then Lemma 12.5 (2) yields $b(v) = b(e) = b(v')$. The Γ-equivariance of this map follows from Lemma 12.4 (2).

Next we show that the map of the lemma is injective. Let $v, v' \in \mathcal{T}_0^{\Gamma\text{-un}}$ and suppose $b(v) = b(v')$. Then the half-lines $H(v)$ and $H(v')$ of Lemma 12.4 (3) agree except finitely many vertices. By Lemma 12.4 (4), each edge in these half-lines is Γ-unstable and thus v and v' are connected with a chain of Γ-unstable edges. Hence we obtain $[v] = [v']$.

For the surjectivity, take any $b \in \mathrm{End}_K(\mathcal{T})$ and let $\{w_n\}_{n \geqslant 0}$ be a half-line which represents b. By Lemma 12.6, we may assume that for any n the edge $f_n = (w_n \to w_{n+1})$ is Γ-unstable with $\mathrm{Stab}_\Gamma(w_n) \subseteq \mathrm{Stab}_\Gamma(w_{n+1})$. Then $\mathrm{Stab}_\Gamma(w_0)$ fixes b and the uniqueness of Lemma 12.4 (1) yields $b = b(w_0)$. This concludes the proof. $\square$

Lemma 12.8. *For any $v \in \mathcal{T}_0^{\Gamma\text{-un}}$, let $e(v)$ be the first edge of the half-line $H(v)$ so that $o(e(v)) = v$. Then the map*

$$\mathcal{T}_0^{\Gamma\text{-un}} \to \mathcal{T}_1^{o,\Gamma\text{-un}}/\{\pm 1\}, \quad v \mapsto [e(v)]$$

is a Γ-equivariant bijection.

Proof. The Γ-equivariance follows from Lemma 12.4 (2) and the uniqueness of $H(v)$. For any $e \in \mathcal{T}_1^{o,\Gamma\text{-un}}$, consider the rational end $b(e)$ of $\mathcal{T}$. For any $w \in \{o(e), t(e)\}$, Lemma 12.5 (2) yields $b(e) = b(w)$. Thus there exists a unique element $v(e) \in \{o(e), t(e)\}$ such that the half-line $H(v(e))$ contains both of $o(e)$ and $t(e)$. Since the definition shows $v(e) = v(-e)$, we obtain a map

$$\mathcal{T}_1^{o,\Gamma\text{-un}}/\{\pm 1\} \to \mathcal{T}_0^{\Gamma\text{-un}}, \quad [e] \mapsto v(e),$$

which gives the inverse of the map $v \mapsto [e(v)]$. □

Definition 12.9. Let $e \in \mathcal{T}_1^o$. We define a subset $\mathrm{src}_\Gamma(e)$ of $\mathcal{T}_1^{o,\Gamma\text{-st}}$ as follows:

- If e is Γ-stable, then $\mathrm{src}_\Gamma(e) = \{e\}$.
- If e is Γ-unstable, then $\mathrm{src}_\Gamma(e)$ consists of Γ-stable edges e' satisfying the conditions below.
  1. There exists a Γ-unstable vertex $v' \in \{o(e'), t(e')\}$ such that the half-line $H(v')$ starting from v' and representing $b(v')$ passes though e. This means that if we write $H(v') = \{w_n\}_{n \geqslant 0}$, then $e = (w_n \to w_{n+1})$ or $e = -(w_n \to w_{n+1})$ for some n.
  2. e' has the same orientation as e with respect to $H(v')$. This means that
  $$e = \begin{cases} (w_n \to w_{n+1}) & (v' = t(e')), \\ -(w_n \to w_{n+1}) & (v' = o(e')). \end{cases}$$

Any element of $\mathrm{src}_\Gamma(e)$ is called a Γ-source of e.

Note that for any Γ-unstable edge e and any $e' \in \mathrm{src}_\Gamma(e)$, the vertex $v' \in \{o(e'), t(e')\}$ satisfying the condition of Definition 12.9 (1) is unique. Indeed, suppose that both of $o(e')$ and $t(e')$ are Γ-unstable. Since e' is Γ-stable, Lemma 12.4 (4) implies that neither $H(o(e'))$ nor $H(t(e'))$ passes through e'. Hence, if both of these half-lines pass through e, then they form a circuit. This is a contradiction.

Moreover, since $\mathrm{Stab}_\Gamma(v')$ fixes $H(v')$, we have $\mathrm{Stab}_\Gamma(v') \subseteq \mathrm{Stab}_\Gamma(e)$ and $b(e) = b(v')$.

From the definition, we have

$$\mathrm{src}_\Gamma(-e) = -\mathrm{src}_\Gamma(e) := \{-e' \mid e' \in \mathrm{src}_\Gamma(e)\}. \tag{12.1}$$

Lemma 12.10 ([Ser], Ch. I, §2.1, Exercise 2)**.** *Let G be a connected locally finite graph containing no injective infinite path. Then G is finite.*

Proof. We may assume $G \neq \emptyset$. Take any vertex $v \in G$. Put $G_0 = \{v\}$. For any integer $i > 0$, let $G_i \subset \mathrm{Vert}(G)$ be the subset consisting of vertices w such that w is adjacent to a vertex in G_{i-1} and $w \notin G_j$ for any $j \leqslant i-1$. Since G is locally finite, each G_i is finite. Since G is connected, we have $G = \bigcup_{i \geqslant 0} G_i$.

For any i and any $w \in G_i$, choose a vertex $g_i(w) \in G_{i-1}$ which is adjacent to w. This gives a map $g_i : G_i \to G_{i-1}$. For any $j > i$, put $g_{j,i} = g_{i+1} \circ \cdots \circ g_j : G_j \to G_i$ and $g_{i,i} = \mathrm{id}$. Then $(G_i, g_{j,i})$ forms an inverse system.

Suppose that G is infinite. Then for any $n \geqslant 0$ there exists $i \geqslant n$ satisfying $G_i \neq \varnothing$. By taking the image of $g_{i,n}$, we see that $G_n \neq \varnothing$ for any $n \geqslant 0$, and [Sta, Lemma 4.21.7] implies $\varprojlim_{i\geqslant 0} G_i \neq \varnothing$. Now $(w_i)_{i\geqslant 0} \in \varprojlim_{i\geqslant 0} G_i$ gives an injective infinite path in G, which is a contradiction. □

Lemma 12.11. *Let $e \in \mathcal{T}_1^o$.*

(1) For any $g \in GL_2(K)$, we have

$$\mathrm{src}_{g\Gamma g^{-1}}(g \circ e) = g \circ \mathrm{src}_\Gamma(e).$$

(2) $\mathrm{src}_\Gamma(e)$ is a finite set.

(3) Suppose that e is Γ-unstable. Let $v \in \{o(e), t(e)\}$ be the farther one from $b(e)$. If $v = o(e)$ (resp. $v = t(e)$), then let $f_1, \ldots, f_q$ be the edges with terminus (resp. origin) v. Then we have

$$\mathrm{src}_\Gamma(e) = \coprod_{i=1}^{q} \mathrm{src}_\Gamma(f_i).$$

Proof. (1) follows from (3.2) and $g \circ (v \to w) = (g \circ v \to g \circ w)$.

For (2), we may assume that e is Γ-unstable. Let $\mathcal{T}_\infty(e)$ be the connected component of $\mathcal{T}_\infty$ containing e. Then Lemma 12.5 (2) yields $b(e) = b(o(e))$ and by Lemma 12.4 it is represented by the half-line $H(o(e)) = \{w_n\}_{n\geqslant 0}$ starting from $o(e)$ and consisting of Γ-unstable edges. Hence Lemma 12.7 implies that $H(o(e))$ represents the unique end in $\mathcal{T}_\infty(e)$.

Now Lemma 12.10 implies that for some $m \geqslant 0$, omitting $\{w_n\}_{n\geqslant m}$ and $\{\pm(w_n \to w_{n+1})\}_{n\geqslant m}$ from $\mathcal{T}_\infty(e)$ defines a finite subgraph. Thus we can find an integer $N \geqslant 0$ such that all the half-lines starting from $o(e)$ or $t(e)$ except those representing $b(e)$ pass through Γ-stable edges before passing through N edges. This shows $|\mathrm{src}_\Gamma(e)| \leqslant 2q^N$.

Let us show (3). Note that if f_i is Γ-unstable, then $b(f_i) = b(v) = b(e)$. From the definition we see $\mathrm{src}_\Gamma(e) = \bigcup_{i=1}^q \mathrm{src}_\Gamma(f_i)$. Suppose $f \in \mathrm{src}_\Gamma(f_i) \cap \mathrm{src}_\Gamma(f_j)$. Then the unique path starting from f and connecting with $H(v)$ passes through both of f_i and f_j, which yields $i = j$ and the union is disjoint. This concludes the proof. □

12.2. **Steinberg module and its resolution.** Also in this subsection, we assume that Γ is p'-torsion free. Let $\mathbb{Z}[\mathbb{P}^1(K)]$ be the free abelian group with basis $\mathbb{P}^1(K)$.

Definition 12.12. Consider the augmentation map

$$\mathrm{aug} : \mathbb{Z}[\mathbb{P}^1(K)] \to \mathbb{Z}, \qquad \sum_{x\in\mathbb{P}^1(K)} n_x[x] \mapsto \sum_{x\in\mathbb{P}^1(K)} n_x.$$

Then we put $\mathrm{St} := \mathrm{Ker}(\mathrm{aug})$ and call it the Steinberg module, so that we have an exact sequence

$$0 \longrightarrow \mathrm{St} \longrightarrow \mathbb{Z}[\mathbb{P}^1(K)] \xrightarrow{\mathrm{aug}} \mathbb{Z} \longrightarrow 0.$$

By Lemma 2.3, the group Γ acts without inversion and thus we can choose an orientation $\mathcal{T}_1^+$ of $\mathcal{T}_1^o$ which is stable under the action of Γ.

Definition 12.13. Put $\mathcal{T}_1^{+,\Gamma\text{-st}} = \mathcal{T}_1^+ \cap \mathcal{T}_1^{o,\Gamma\text{-st}}$ and

$$S_0 := \mathcal{T}_0^{\Gamma\text{-st}}, \quad S_1 := \mathcal{T}_1^{+,\Gamma\text{-st}}.$$

Lemma 12.14. *The group Γ acts freely on S_0 and S_1 from the left via the action $\circ$.*

Proof. By (3.2), the group Γ acts on $S_0 = \mathcal{T}_0^{\Gamma\text{-st}}$ and $\mathcal{T}_1^{o,\Gamma\text{-st}}$ from the left via $\circ$. Since the orientation $\mathcal{T}_1^+$ is Γ-stable, it also acts on S_1. For any Γ-stable simplex s we have $\mathrm{Stab}_\Gamma(s) = \{\mathrm{id}\}$ and the freeness of the action follows. $\square$

Definition 12.15. Put

$$l_0 = |\Gamma\backslash S_0|, \quad l_1 = |\Gamma\backslash S_1|.$$

Lemma 12.16. *The cardinality l_i is finite for $i = 0, 1$.*

Proof. Take $g \in GL_2(K)$ satisfying $g\Gamma g^{-1} \subseteq GL_2(A)$. By (3.2), the map $s \mapsto g \circ s$ induces bijections

$$\Gamma\backslash\mathcal{T}_0^{\Gamma\text{-st}} \to g\Gamma g^{-1}\backslash\mathcal{T}_0^{g\Gamma g^{-1}\text{-st}}, \quad \Gamma\backslash\mathcal{T}_1^{o,\Gamma\text{-st}}/\{\pm 1\} \to g\Gamma g^{-1}\backslash\mathcal{T}_1^{o,g\Gamma g^{-1}\text{-st}}/\{\pm 1\}.$$

Thus we may assume $\Gamma \subseteq GL_2(A)$.

By Lemma 3.12 and Lemma 3.15, we see that the quotient graph $\Gamma\backslash\mathcal{T}$ is the union of a finite graph and the image of finitely many rational ends in $\mathcal{T}$. Hence Lemma 12.6 implies that any Γ-stable simplex is Γ-equivalent to a simplex which lies in the finite graph obtained from $\Gamma\backslash\mathcal{T}$ by cutting off injective infinite paths. This concludes the proof. $\square$

Definition 12.17. Define

$$L_0 := \mathbb{Z}[S_0], \quad L_1 := \mathbb{Z}[S_1].$$

By the action induced by $\circ$, they are considered as left $\mathbb{Z}[\Gamma]$-modules. Then the left $\mathbb{Z}[\Gamma]$-module L_i is free of rank l_i for $i = 0, 1$.

Though the definition of L_1 depends on the choice of a Γ-stable orientation $\mathcal{T}_1^+$ of $\mathcal{T}$, we have the following description of L_1 which is independent of the choice.

Lemma 12.18. *The natural map*

$$L_1 = \mathbb{Z}[S_1] \to \mathbb{Z}[\mathcal{T}_1^{o,\Gamma\text{-st}}]/\langle [e] + [-e] \mid e \in \mathcal{T}_1^{o,\Gamma\text{-st}} \rangle, \quad [e] \mapsto [e]$$

is an isomorphism of left $\mathbb{Z}[\Gamma]$*-modules.*

Proof. The map of the lemma is $\mathbb{Z}[\Gamma]$-linear. It is enough to construct its inverse as a morphism of $\mathbb{Z}$-modules. Define a $\mathbb{Z}$-linear map

$$F : \mathbb{Z}[\mathcal{T}_1^{o,\Gamma\text{-st}}] \to \mathbb{Z}[S_1]$$

by $F([e]) = [e]$ if $e \in \mathcal{T}_1^+$ and $F([e]) = -[-e]$ if not. Then we have $F([e] + [-e]) = 0$ for any $e \in \mathcal{T}_1^{o,\Gamma\text{-st}}$ and the map F induces a $\mathbb{Z}$-linear map

$$\mathbb{Z}[\mathcal{T}_1^{o,\Gamma\text{-st}}]/\langle [e] + [-e] \mid e \in \mathcal{T}_1^{o,\Gamma\text{-st}} \rangle \to \mathbb{Z}[S_1],$$

which gives the inverse of the map of the lemma. □

The graph $\mathcal{T}$ defines a simplicial complex X whose set of zero-dimensional simplices is $\mathcal{T}_0$ and that of one-dimensional simplices is $\mathcal{T}_1^+$. Similarly, $\mathcal{T}_\infty$ defines a simplicial complex X_∞. We denote the group of i-dimensional chains of them by $C_i(X)$ and $C_i(X_\infty)$. Put

$$C_i(X, X_\infty) := C_i(X)/C_i(X_\infty).$$

Then we have a natural isomorphism of left $\mathbb{Z}[\Gamma]$-modules

$$L_i \to C_i(X, X_\infty).$$

Proposition 12.19. *The left* $\mathbb{Z}[\Gamma]$*-modules* $\mathrm{St} \oplus L_0$ *and* L_1 *are isomorphic. In particular, we have* $l_1 \geqslant l_0$ *and the left* $\mathbb{Z}[\Gamma]$*-module* St *is finitely generated and projective.*

Proof. The long exact sequence of relative homology groups gives an exact sequence of left $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow H_1(X_\infty) \longrightarrow H_1(X) \longrightarrow H_1(X, X_\infty) \longrightarrow H_0(X_\infty)$$

$$\longrightarrow H_0(X) \longrightarrow H_0(X, X_\infty) \longrightarrow 0.$$

Since $\mathcal{T}$ is a connected tree, we have $H_1(X) = 0$ and $H_0(X) = \mathbb{Z}$. Since the map $H_0(X_\infty) \to H_0(X)$ can be identified with the map sending a connected component of $\mathcal{T}_\infty$ to that of $\mathcal{T}$, it is surjective and $H_0(X, X_\infty) = 0$.

Thus we have an exact sequence of left $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow H_1(X, X_\infty) \longrightarrow H_0(X_\infty) \xrightarrow{\text{aug}} \mathbb{Z} \longrightarrow 0,$$

where the map aug sends any connected component of $\mathcal{T}_\infty$ to one. By Lemma 12.7, this map is identified with the augmentation map $\mathbb{Z}[\mathbb{P}^1(K)] \to \mathbb{Z}$ and we obtain an isomorphism of left $\mathbb{Z}[\Gamma]$-modules

$$H_1(X, X_\infty) \to \mathrm{St} = \mathrm{Ker}(\mathrm{aug}).$$

On the other hand, by the definition of relative homology groups we have an exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow H_1(X, X_\infty) \longrightarrow C_1(X, X_\infty) \longrightarrow C_0(X, X_\infty) \longrightarrow H_0(X, X_\infty) = 0,$$

which yields an exact sequence of $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow \mathrm{St} \longrightarrow L_1 \xrightarrow{\partial} L_0 \longrightarrow 0. \tag{12.2}$$

Here the map $\partial : L_1 \to L_0$ is given by $\partial([e]) = [t(e)] - [o(e)]$, where we put $[v] = 0$ when $v \in \mathcal{T}_0^{\Gamma\text{-un}}$. Since the left $\mathbb{Z}[\Gamma]$-module L_i is free of rank l_i, the exact sequence (12.2) splits. Hence the $\mathbb{Z}[\Gamma]$-module St is finitely generated and projective. By tensoring the augmentation map $\mathbb{Z}[\Gamma] \to \mathbb{Z}$ from the left to (12.2), we obtain $l_1 \geqslant l_0$. This concludes the proof. □

Lemma 12.20. *Let $\chi(\Gamma)$ be the Euler–Poincaré characteristic of Γ, as in Definition 3.17. Then we have*

$$\chi(\Gamma) = l_0 - l_1.$$

Proof. Recall that $\chi(\Gamma)$ is defined as the absolutely convergent series

$$\chi(\Gamma) = \sum_{v \in \Gamma\backslash\mathcal{T}_0} \frac{1}{|\mathrm{Stab}_\Gamma(v)|} - \sum_{e \in \Gamma\backslash\mathcal{T}_1^o/\{\pm1\}} \frac{1}{|\mathrm{Stab}_\Gamma(e)|}.$$

Put

$$\chi_{\mathrm{st}}(\Gamma) = \sum_{v \in \Gamma\backslash\mathcal{T}_0^{\Gamma\text{-st}}} \frac{1}{|\mathrm{Stab}_\Gamma(v)|} - \sum_{e \in \Gamma\backslash\mathcal{T}_1^{o,\Gamma\text{-st}}/\{\pm1\}} \frac{1}{|\mathrm{Stab}_\Gamma(e)|},$$

$$\chi_{\mathrm{un}}(\Gamma) = \sum_{v \in \Gamma\backslash\mathcal{T}_0^{\Gamma\text{-un}}} \frac{1}{|\mathrm{Stab}_\Gamma(v)|} - \sum_{e \in \Gamma\backslash\mathcal{T}_1^{o,\Gamma\text{-un}}/\{\pm1\}} \frac{1}{|\mathrm{Stab}_\Gamma(e)|},$$

so that they are also absolutely convergent and $\chi(\Gamma) = \chi_{\mathrm{st}}(\Gamma) + \chi_{\mathrm{un}}(\Gamma)$.

For $\chi_{\mathrm{st}}(\Gamma)$, the stabilizer subgroups are all trivial and thus

$$\chi_{\mathrm{st}}(\Gamma) = |\Gamma\backslash\mathcal{T}_0^{\Gamma\text{-st}}| - |\Gamma\backslash\mathcal{T}_1^{o,\Gamma\text{-st}}/\{\pm1\}| = l_0 - l_1.$$

Thus we are reduced to showing $\chi_{\mathrm{un}}(\Gamma) = 0$.

Consider the Γ-equivariant bijection

$$\mathcal{T}_0^{\Gamma\text{-un}} \to \mathcal{T}_1^{o,\Gamma\text{-un}}/\{\pm1\}, \quad v \mapsto [e(v)]$$

of Lemma 12.8. By definition, we have $o(e(v)) = v$ and thus $\mathrm{Stab}_\Gamma(e(v)) \subseteq \mathrm{Stab}_\Gamma(v)$. On the other hand, by Lemma 12.4 (4) the group $\mathrm{Stab}_\Gamma(v)$ fixes the half-line $H(v)$ and thus $\mathrm{Stab}_\Gamma(v) \subseteq \mathrm{Stab}_\Gamma(e(v))$. Hence we obtain $\mathrm{Stab}_\Gamma(v) = \mathrm{Stab}_\Gamma(e(v))$ and

$$\chi_{\mathrm{un}}(\Gamma) = \sum_{v \in \Gamma\backslash \mathcal{T}_0^{\Gamma\text{-un}}} \left(\frac{1}{|\mathrm{Stab}_\Gamma(v)|} - \frac{1}{|\mathrm{Stab}_\Gamma(e(v))|} \right) = 0.$$

This concludes the proof. □

12.3. **Euler–Poincaré characteristic and group homology.** Let Γ be an arithmetic subgroup of $GL_2(K)$. For any left $\mathbb{Z}[\Gamma]$-module M, we denote by M_Γ the module of Γ-coinvariants of M. Then the group homology $H_i(\Gamma, M)$ is the i-th left derived functor of $M \mapsto M_\Gamma$.

Since $\mathcal{T}$ is a tree, the boundary map $\partial : C_1(X) \to C_0(X)$ gives an exact sequence of $\mathbb{Z}[\Gamma]$-module

$$0 \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

For $i = 0, 1$, put $C_i(X)_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} C_i(X)$. Then we have the long exact sequence of group homology
(12.3)

$$\cdots \longrightarrow H_i(\Gamma, C_1(X)_\mathbb{Q}) \longrightarrow H_i(\Gamma, C_0(X)_\mathbb{Q}) \longrightarrow H_i(\Gamma, \mathbb{Q})$$

$$\longrightarrow H_{i-1}(\Gamma, C_1(X)_\mathbb{Q}) \longrightarrow \cdots.$$

Lemma 12.21. *Let Γ be an arithmetic subgroup of $GL_2(K)$. Then we have an isomorphism of $\mathbb{Q}$-vector spaces*

$$H_1(\Gamma, \mathbb{Q}) \simeq H_1(\Gamma\backslash X, \mathbb{Q}).$$

Proof. Let Σ_0 and Σ_1 be complete sets of representatives of $\Gamma\backslash\mathcal{T}_0$ and $\Gamma\backslash\mathcal{T}_1^+$, respectively. Then for $i = 0, 1$ we have Γ-equivariant isomorphisms

$$C_i(X) \to \bigoplus_{s \in \Sigma_i} \mathbb{Z}[\Gamma/\mathrm{Stab}_\Gamma(s)], \quad C_i(X)_\mathbb{Q} \to \bigoplus_{s \in \Sigma_i} \mathbb{Q}[\Gamma/\mathrm{Stab}_\Gamma(s)].$$

By Shapiro's lemma [Bro, Ch. III, Proposition 6.2], we also have an isomorphism

$$H_i(\Gamma, \mathbb{Q}[\Gamma/\mathrm{Stab}_\Gamma(s)]) \simeq H_i(\mathrm{Stab}_\Gamma(s), \mathbb{Q}).$$

By Lemma 3.6, the group $\mathrm{Stab}_\Gamma(s)$ is finite. Then [Bro, Ch. III, Proposition 9.5 (ii)] implies $H_i(\mathrm{Stab}_\Gamma(s), \mathbb{Q}) = 0$ for any $i > 0$. Hence (12.3) yields an exact sequence

$$0 \longrightarrow H_1(\Gamma, \mathbb{Q}) \longrightarrow H_0(\Gamma, C_1(X)_{\mathbb{Q}}) \longrightarrow H_0(\Gamma, C_0(X)_{\mathbb{Q}}).$$

Since for $i = 0, 1$ we have an isomorphism

$$H_0(\Gamma, C_i(X)_{\mathbb{Q}}) = (C_i(X)_{\mathbb{Q}})_\Gamma \simeq C_i(\Gamma\backslash X)_{\mathbb{Q}}$$

compatible with boundary maps, the lemma follows. □

Let Σ be a complete set of representatives of $\mathrm{Cusps}(\Gamma) = \Gamma\backslash\mathbb{P}^1(K)$. Then we have an isomorphism of left $\mathbb{Z}[\Gamma]$-modules

$$\mathbb{Z}[\mathbb{P}^1(K)] \simeq \bigoplus_{\sigma\in\Sigma} \mathbb{Z}[\Gamma/\mathrm{Stab}_\Gamma(\sigma)],$$

which yields an exact sequence of left $\mathbb{Z}[\Gamma]$-modules

$$0 \longrightarrow \mathrm{St} \longrightarrow \bigoplus_{\sigma\in\Sigma} \mathbb{Z}[\Gamma/\mathrm{Stab}_\Gamma(\sigma)] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Hence we have the long exact sequence of group homology

(12.4)
$$\cdots \longrightarrow \bigoplus_{\sigma\in\Sigma} H_i(\mathrm{Stab}_\Gamma(\sigma), \mathbb{Q}) \longrightarrow H_i(\Gamma, \mathbb{Q}) \longrightarrow H_{i-1}(\Gamma, \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St})$$

$$\longrightarrow \bigoplus_{\sigma\in\Sigma} H_{i-1}(\mathrm{Stab}_\Gamma(\sigma), \mathbb{Q}) \longrightarrow \cdots.$$

Lemma 12.22. *Let Γ be an arithmetic subgroup of $GL_2(K)$ which is p'-torsion free. For any $i > 0$, we have*

$$H_i(\Gamma, \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St}) = 0, \quad \dim_{\mathbb{Q}}(H_0(\Gamma, \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St})) = -\chi(\Gamma).$$

Proof. Since $H_i(\Gamma, \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St}) \simeq \mathrm{Tor}_i^{\mathbb{Z}[\Gamma]}(\mathbb{Q}, \mathrm{St})$, the first assertion follows from Proposition 12.19. Applying the functor $\mathbb{Q}\otimes_{\mathbb{Z}[\Gamma]} -$ with (12.2) and using the isomorphism

$$H_0(\Gamma, \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St}) = (\mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St})_\Gamma \simeq \mathbb{Q}\otimes_{\mathbb{Z}[\Gamma]} \mathrm{St},$$

we obtain an exact sequence of $\mathbb{Q}$-vector spaces

$$0 \longrightarrow H_0(\Gamma, \mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{St}) \longrightarrow \mathbb{Q}^{l_1} \longrightarrow \mathbb{Q}^{l_0} \longrightarrow 0.$$

Thus Lemma 12.20 concludes the proof. □

Lemma 12.23. *Let $\sigma \in \mathbb{P}^1(K)$. Then*

$$\mathbb{Q}\otimes_{\mathbb{Z}} \mathrm{Stab}_\Gamma(\sigma)^{\mathrm{ab}} = 0.$$

Proof. Replacing Γ by its conjugate, we may assume that Γ is a congruence subgroup. Since we have $g \circ \infty = \sigma$ with some $g \in GL_2(A)$, replacing Γ by $g\Gamma g^{-1}$ we may also assume $\sigma = \infty$. Then we have

$$\mathrm{Stab}_\Gamma(\infty) \subseteq \left\{ \gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \,\middle|\, a, d \in \mathbb{F}_q^\times,\ b \in A \right\}.$$

Suppose $\gamma \in \mathrm{Stab}_\Gamma(\infty)$. If $a \neq d$, then as in the proof of Lemma 6.50 we have $\gamma^{q-1} = \mathrm{id}$. Otherwise

$$\gamma^{p(q-1)} = \left(a \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right)^{p(q-1)} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}^{p(q-1)} = \mathrm{id}.$$

Thus the module $\mathrm{Stab}_\Gamma(\infty)^{\mathrm{ab}}$ is torsion. This concludes the proof. □

Proposition 12.24 ([Ser], Ch. II, §2.9, Exercise 2)**.** *Let Γ be an arithmetic subgroup of $GL_2(K)$ which is p'-torsion free. Let*

$$g := \dim_{\mathbb{Q}}(H_1(\Gamma\backslash X, \mathbb{Q})), \quad h := |\mathrm{Cusps}(\Gamma)|.$$

Then we have $\chi(\Gamma) = -(g + h - 1)$.

Proof. By Lemma 12.23 and the universal coefficient theorem [Bro, Ch. III, §1, Exercise 3], for any $\sigma \in \Sigma$ we have

$$H_1(\mathrm{Stab}_\Gamma(\sigma), \mathbb{Q}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Stab}_\Gamma(\sigma)^{\mathrm{ab}} = 0, \quad H_0(\mathrm{Stab}_\Gamma(\sigma), \mathbb{Q}) = \mathbb{Q}.$$

Note that $h = |\Sigma|$ and $H_0(\Gamma, \mathbb{Q}) = \mathbb{Q}$. By Lemma 12.21 and Lemma 12.22, the sequence (12.4) yields an exact sequence of $\mathbb{Q}$-vector spaces

$$0 \longrightarrow H_1(\Gamma\backslash X, \mathbb{Q}) \longrightarrow \mathbb{Q}^{-\chi(\Gamma)} \longrightarrow \mathbb{Q}^h \longrightarrow \mathbb{Q} \longrightarrow 0,$$

from which the proposition follows. □

12.4. **Description of Drinfeld cuspforms via harmonic cocycles.**

Definition 12.25. Let $k \geqslant 2$ be an integer. Suppose that Γ is p'-torsion free. We denote by $C_k^{\mathrm{st,har}}(\Gamma)$ the $\mathbb{C}_\infty$-vector space consisting of maps $c : \mathcal{T}_1^{o,\Gamma\text{-st}} \to V_k(\mathbb{C}_\infty)$ satisfying the following conditions.

(1) For any $v \in \mathcal{T}_0^{\Gamma\text{-st}}$, we have
$$\sum_{e \in \mathcal{T}_1^o,\ t(e)=v} c(e) = 0.$$
Note that the assumption $v \in \mathcal{T}_0^{\Gamma\text{-st}}$ forces e in the sum to be Γ-stable.

(2) For any $e \in \mathcal{T}_1^{o,\Gamma\text{-st}}$, we have $c(-e) = -c(e)$.

(3) For any $\gamma \in \Gamma$ and $e \in \mathcal{T}_0^{\Gamma\text{-st}}$, we have $\gamma \circ c(e) = c(\gamma \circ e)$.

Moreover, we denote by $C_k^{\mathrm{st},\pm}(\Gamma)$ the $\mathbb{C}_\infty$-vector space consisting of maps $c : \mathcal{T}_1^{o,\Gamma\text{-st}} \to V_k(\mathbb{C}_\infty)$ satisfying the conditions (2) and (3).

Lemma 12.26. *Let Λ_1 be a complete set of representatives of $\Gamma\backslash S_1 = \Gamma\backslash\mathcal{T}_1^{+,\Gamma\text{-st}}$. For any $e \in \mathcal{T}_1^{o,\Gamma\text{-st}}$, there exists a unique triple $(\varepsilon_e, \gamma_e, r(e)) \in \{\pm 1\} \times \Gamma \times \Lambda_1$ satisfying $e = \varepsilon_e\gamma_e \circ r(e)$. Moreover, for any $\delta \in \Gamma$ we have*

$$(\varepsilon_{-e}, \gamma_{-e}, r(-e)) = (-\varepsilon_e, \gamma_e, r(e)), \quad (\varepsilon_{\delta\circ e}, \gamma_{\delta\circ e}, r(\delta\circ e)) = (\varepsilon_e, \delta\gamma_e, r(e)).$$

Proof. Since $\mathcal{T}_1^{+}$ is an orientation, we can find $\varepsilon_e \in \{\pm 1\}$ satisfying $\varepsilon_e e \in \mathcal{T}_1^{+,\Gamma\text{-st}}$. This yields the existence of such a triple.

For the uniqueness, suppose triples $(\varepsilon_e, \gamma_e, r(e))$ and $(\varepsilon'_e, \gamma'_e, r'(e))$ satisfy

$$e = \varepsilon_e\gamma_e \circ r(e) = \varepsilon'_e\gamma'_e \circ r'(e).$$

Since $\mathcal{T}_1^{+}$ is a Γ-stable orientation, both of $\gamma_e \circ r(e)$ and $\gamma'_e \circ r'(e)$ lie in $\mathcal{T}_1^{+}$ and thus $\varepsilon_e = \varepsilon'_e$. Since Λ_1 is a complete set of representatives, we have $r(e) = r'(e)$ and $\gamma_e^{-1}\gamma'_e \circ r(e) = r(e)$. Since $r(e)$ is Γ-stable, we obtain $\gamma_e = \gamma'_e$. The last assertion follows from the uniqueness. $\square$

Lemma 12.27. *Let Λ_1 be a complete set of representatives of $\Gamma\backslash S_1$. Then we have a $\mathbb{C}_\infty$-linear isomorphism*

$$C_k^{\mathrm{st},\pm}(\Gamma) \to \bigoplus_{e\in\Lambda_1} V_k(\mathbb{C}_\infty), \quad c \mapsto (c(e))_{e\in\Lambda_1}.$$

Proof. By Lemma 12.26, for any $f \in \mathcal{T}_1^{o,\Gamma\text{-st}}$, we can find a unique triple $(\varepsilon_f, \gamma_f, r(f)) \in \{\pm 1\} \times \Gamma \times \Lambda_1$ satisfying $f = \varepsilon_f\gamma_f \circ r(f)$. Then the map

$$\bigoplus_{e\in\Lambda_1} V_k(\mathbb{C}_\infty) \to C_k^{\mathrm{st},\pm}(\Gamma), \quad (\omega_e)_{e\in\Lambda_1} \mapsto (f \mapsto \varepsilon_f\gamma_f \circ \omega_{r(f)})$$

is well-defined and gives the inverse of the map of the lemma. $\square$

Lemma 12.28. *Suppose that Γ is p'-torsion free. Then*

$$\dim_{\mathbb{C}_\infty}(C_k^{\mathrm{st},\mathrm{har}}(\Gamma)) \geqslant (k-1)(l_1 - l_0).$$

Proof. Let Λ_i be a complete set of representatives of $\Gamma\backslash S_i$. Consider the basis $\{X^{k-2-l}Y^l\}_{l=0,\dots,k-2}$ of $H_{k-2}(\mathbb{C}_\infty)$ and its dual basis $\{(X^{k-2-l}Y^l)^\vee\}_{l=0,\dots,k-2}$ of $V_k(\mathbb{C}_\infty)$. For any $c \in C_k^{\mathrm{st},\pm}(\Gamma)$ and $e \in \Lambda_1$, write

$$c(e) = \sum_{l=0}^{k-2} a_{e,l}(c)(X^{k-2-l}Y^l)^\vee, \quad a_{e,l}(c) \in \mathbb{C}_\infty.$$

By Lemma 12.27, we have a $\mathbb{C}_\infty$-linear isomorphism

$$C_k^{\mathrm{st},\pm}(\Gamma) \to \bigoplus_{e\in\Lambda_1}\bigoplus_{l=0}^{k-2} \mathbb{C}_\infty, \quad c \mapsto (a_{e,l}(c))_{e,l}. \tag{12.5}$$

Moreover, by Lemma 12.26, for any $e \in \mathcal{T}_1^{o,\Gamma\text{-st}}$ we can uniquely write

$$e = \varepsilon_e \gamma_e \circ r(e), \quad (\varepsilon_e, \gamma_e, r(e)) \in \{\pm 1\} \times \Gamma \times \Lambda_1.$$

Fix $v \in \Lambda_0$. Then we have

$$\sum_{t(e)=v} c(e) = 0 \quad \Leftrightarrow \quad \sum_{t(e)=v} \varepsilon_e(\gamma_e \circ c(r(e)))(X^{k-2-l}Y^l) = 0 \text{ for any } l.$$

Thus the condition that $c \in C_k^{\mathrm{st},\pm}(\Gamma)$ lies in $C_k^{\mathrm{st,har}}(\Gamma)$ is identified, via the isomorphism (12.5), with $(k-1)l_0$ linear relations on $\mathbb{C}_\infty^{(k-1)l_1}$. Thus the $\mathbb{C}_\infty$-vector space $C_k^{\mathrm{st,har}}(\Gamma)$ is isomorphic to the null space

$$\{x \in \mathbb{C}_\infty^{(k-1)l_1} \mid Bx = 0\}$$

for some $(k-1)l_0 \times (k-1)l_1$ matrix B with entries in $\mathbb{C}_\infty$. Now Proposition 12.19 yields $l_1 \geqslant l_0$ and

$$\dim_{\mathbb{C}_\infty}(C_k^{\mathrm{har,st}}(\Gamma)) = (k-1)l_1 - \mathrm{rank}(B) \geqslant (k-1)l_1 - (k-1)l_0.$$

This concludes the proof. □

Lemma 12.29. *Suppose that Γ is p'-torsion free. Then the restriction to $\mathcal{T}_1^{o,\Gamma\text{-st}}$ gives a $\mathbb{C}_\infty$-linear isomorphism*

$$C_k^{\mathrm{har}}(\Gamma) \to C_k^{\mathrm{st,har}}(\Gamma), \quad c \mapsto c|_{\mathcal{T}_1^{o,\Gamma\text{-st}}}.$$

In particular, we have

$$\dim_{\mathbb{C}_\infty}(C_k^{\mathrm{har}}(\Gamma)) \geqslant (k-1)(l_1 - l_0).$$

Proof. From the definition of Γ-sources, we see that for any $c \in C_k^{\mathrm{har}}(\Gamma)$ and any $e \in \mathcal{T}_1^o$ we have

$$c(e) = \sum_{e' \in \mathrm{src}_\Gamma(e)} c(e').$$

Thus the harmonic cocycle c is determined by its restriction to Γ-stable edges, and the map of the lemma is injective.

For the surjectivity, take any $c \in C_k^{\mathrm{st,har}}(\Gamma)$. We define a map $\tilde{c} : \mathcal{T}_1^o \to V_k(\mathbb{C}_\infty)$ by

$$\tilde{c}(e) := \sum_{e' \in \mathrm{src}_\Gamma(e)} c(e').$$

By Lemma 12.11 (2), it is well-defined and its restriction to $\mathcal{T}_1^{o,\Gamma\text{-st}}$ is c. By Lemma 12.11 (1), the map $\tilde{c}$ is Γ-equivariant. (12.1) yields $\tilde{c}(-e) = -\tilde{c}(e)$.

Let us show that $\tilde{c}$ is harmonic at any vertex $v \in \mathcal{T}_0$. We may assume that v is Γ-unstable. Consider the half-line $H(v)$ of Lemma 12.4 (3). Then Lemma 12.4 (4) shows that the first edge e of $H(v)$ is Γ-unstable and satisfies $o(e) = v$. Let $f_1, \dots, f_q$ be the edges with terminus v. By Lemma 12.11 (3), we have

$$\tilde{c}(e) = \sum_{e' \in \mathrm{src}_\Gamma(e)} c(e') = \sum_{i=1}^{q} \sum_{e' \in \mathrm{src}_\Gamma(f_i)} c(e') = \sum_{i=1}^{q} \tilde{c}(f_i),$$

which shows the harmonicity of $\tilde{c}$ at the vertex v. Thus the map of the lemma is surjective. The last assertion follows from Lemma 12.28. □

Theorem 12.30. *Let Γ be an arithmetic subgroup of $GL_2(K)$ and let $k \geqslant 2$ be an integer. Then the map*

$$C_k^{\mathrm{har}}(\Gamma) \to S_k(\Gamma), \quad c \mapsto F_c$$

of Proposition 11.20 is a $\mathbb{C}_\infty$-linear isomorphism with the inverse $f \mapsto$ $\mathrm{Res}(f)$.

Proof. By Proposition 11.20, the theorem is equivalent to the inequality

$$\dim_{\mathbb{C}_\infty}(C_k^{\mathrm{har}}(\Gamma)) \geqslant \dim_{\mathbb{C}_\infty}(S_k(\Gamma)). \tag{12.6}$$

For any $\nu \in GL_2(K)$, we have an isomorphism

$$C_k^{\mathrm{har}}(\Gamma) \to C_k^{\mathrm{har}}(\nu^{-1}\Gamma\nu), \quad c \mapsto {}^{\nu^{-1}}c : (e \mapsto \nu^{-1} \circ c(\nu \circ e)).$$

By Lemma 6.62, replacing Γ with its conjugate, to show the inequality (12.6) we may assume $\Gamma \subseteq GL_2(A)$. Then we have $\Gamma(\mathfrak{n}) \lhd \Gamma$ for some nonzero ideal $\mathfrak{n} \subseteq A$.

Note that the group $\Gamma/\Gamma(\mathfrak{n})$ acts on $C_k^{\mathrm{har}}(\Gamma(\mathfrak{n}))$ from the right via $c \mapsto {}^{\gamma^{-1}}c$ for any $\gamma \in \Gamma$ and

$$C_k^{\mathrm{har}}(\Gamma) = C_k^{\mathrm{har}}(\Gamma(\mathfrak{n}))^{\Gamma/\Gamma(\mathfrak{n})}.$$

If the theorem holds for $\Gamma(\mathfrak{n})$, then Lemma 11.15 and Lemma 6.63 yield $\dim_{\mathbb{C}_\infty}(C_k^{\mathrm{har}}(\Gamma)) = \dim_{\mathbb{C}_\infty}(S_k(\Gamma))$. Hence the theorem also holds for Γ. Therefore, we reduce ourselves to showing the theorem for $\Gamma(\mathfrak{n})$, which is p'-torsion free.

For this, by Lemma 12.29 and Lemma 12.20 we have

$$\dim_{\mathbb{C}_\infty}(C_k^{\mathrm{har}}(\Gamma(\mathfrak{n}))) \geqslant (k-1)(l_1 - l_0) = -(k-1)\chi(\Gamma(\mathfrak{n})).$$

On the other hand, Lemma 3.18 and (3.8) yield

$$-\chi(\Gamma(\mathfrak{n})) = \frac{[GL_2(A) : \Gamma(\mathfrak{n})]}{(q-1)^2(q+1)}.$$

Let $g_{\mathfrak{n}}$ be the genus of the compactification $X(\mathfrak{n})$ of $\Gamma(\mathfrak{n})\backslash\Omega$ and let $h = |\mathrm{Cusps}(\Gamma(\mathfrak{n}))|$. Then [Gek1, Ch. VII, Theorem 5.11] gives

$$g_{\mathfrak{n}} = 1 - \chi(\Gamma(\mathfrak{n})) - h.$$

By [Gos1, Corollary 1.81], we have an invertible sheaf ω on the projective smooth curve $X(\mathfrak{n})$ such that $\deg(\omega) = g_{\mathfrak{n}} - 1 + h$ and there exists a natural isomorphism

$$S_k(\Gamma(\mathfrak{n})) \to H^0(X(\mathfrak{n}), \omega^k(-\mathrm{Cusps})).$$

Since $k \geqslant 2$ and $h \geqslant 1$, we have

$$\deg(\omega^k(-\mathrm{Cusps})) = k(g_{\mathfrak{n}} - 1 + h) - h > 2g_{\mathfrak{n}} - 2.$$

Now the Riemann–Roch theorem yields

$$\begin{aligned}\dim_{\mathbb{C}_\infty}(S_k(\Gamma(\mathfrak{n}))) &= k(g_{\mathfrak{n}} - 1 + h) - h + 1 - g_{\mathfrak{n}} \\ &= (k-1)(g_{\mathfrak{n}} - 1 + h) = -(k-1)\chi(\Gamma(\mathfrak{n})),\end{aligned}$$

from which the theorem follows. □

Remark 12.31. The use of Gekeler's genus formula can be bypassed by using Proposition 12.24 and the fact that the graph obtained from $\Gamma\backslash\mathcal{T}$ by cutting off all ends agrees with the dual graph of the semistable reduction of the compactification $X(\Gamma)$ of $\Gamma\backslash\Omega$ and the first Betti number of the dual graph gives the genus of $X(\Gamma)$. A construction of the compactification is explained in [Böc, §3.7]. On the other hand, in order to construct the Hodge bundle ω we need the theory of Drinfeld modules and Tate–Drinfeld modules. I hope to add these topics to the notes when I have time.

12.5. **Steinberg module and harmonic cocycles.** Let Γ be an arithmetic subgroup of $GL_2(K)$ which is p'-torsion free. For $i = 0, 1$, let S_i be the set of Definition 12.13 and let $L_i = \mathbb{Z}[S_i]$ as in Definition 12.17. Let Λ_i be a complete set of representatives of $\Gamma\backslash S_i$.

For any left $\mathbb{Z}[\Gamma]$-module $M \in \{L_0, L_1, \mathrm{St}\}$, we consider M as a right $\mathbb{Z}[\Gamma]$-module by

$$m \cdot \gamma := \gamma^{-1} \circ m, \quad \gamma \in \Gamma,\ m \in M,$$

so that for any integer $k \geqslant 2$, we can form the tensor product $M \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty)$. Then for $i = 0, 1$, the right $\mathbb{Z}[\Gamma]$-module L_i is also free. Hence (12.2) induces an exact sequence of $\mathbb{C}_\infty$-vector spaces
(12.7)

$$0 \longrightarrow \mathrm{St} \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \longrightarrow L_1 \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \xrightarrow{\partial\otimes 1} L_0 \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty) \longrightarrow 0.$$

Definition 12.32. For any $k \geqslant 2$, define a $\mathbb{C}_\infty$-linear map $\phi_\Gamma^{\mathrm{st}}$ by

$$\phi_\Gamma^{\mathrm{st}} : C_k^{\mathrm{st},\pm}(\Gamma) \to L_1 \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty), \quad c \mapsto \sum_{e \in \Lambda_1} [e] \otimes c(e).$$

Then Lemma 12.27 implies that $\phi_\Gamma^{\mathrm{st}}$ is an isomorphism. For any $\gamma \in \Gamma$ and $e \in S_1 = \mathcal{T}_1^{+,\Gamma\text{-st}}$, in the module $L_1 \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty)$ we have

$$[\gamma \circ e] \otimes c(\gamma \circ e) = [e] \cdot \gamma^{-1} \otimes \gamma \circ c(e) = [e] \otimes c(e).$$

Hence the map $\phi_\Gamma^{\mathrm{st}}$ is independent of the choice of Λ_1.

Lemma 12.33. *The map $\phi_\Gamma^{\mathrm{st}}$ induces a $\mathbb{C}_\infty$-linear isomorphism*

$$C_k^{\mathrm{st,har}}(\Gamma) \to \mathrm{St} \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty).$$

Proof. Note that for $i = 0, 1$, the freeness of the right $\mathbb{Z}[\Gamma]$-module L_i implies that the $\mathbb{C}_\infty$-linear map

$$\bigoplus_{s \in \Lambda_i} V_k(\mathbb{C}_\infty) \to L_i \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty), \quad (\omega_s)_{s \in \Lambda_i} \mapsto \sum_{s \in \Lambda_i} [s] \otimes \omega_s$$

is an isomorphism. Let $c \in C_k^{\mathrm{st},\pm}(\Gamma)$ and write

$$(\partial \otimes 1)(\phi_\Gamma^{\mathrm{st}}(c)) = \sum_{v \in \Lambda_0} [v] \otimes \omega_{c,v}$$

with some $\omega_{c,v} \in V_k(\mathbb{C}_\infty)$. By the exact sequence (12.7), it is enough to show that $\omega_{c,v} = 0$ for any $v \in \Lambda_0$ if and only if $c \in C_k^{\mathrm{st,har}}(\Gamma)$.

Take any $v \in \Lambda_0$. Let $\Lambda(v) = \{e \in \mathcal{T}_1^o \mid t(e) = v\}$. Since $\mathcal{T}_1^+$ is an orientation, for any $e \in \Lambda(v)$ there exists a unique $\varepsilon_e \in \{\pm 1\}$ satisfying $\varepsilon_e e \in \mathcal{T}_1^+$. Put

$$\Lambda(v)^+ := \{\varepsilon_e e \mid e \in \Lambda(v)\}.$$

Since v is Γ-stable, any edge in $\Lambda(v)$ is Γ-stable. Moreover, for any $e, e' \in \Lambda(v)$ we have $e' \notin \Gamma e$, and Lemma 2.3 also implies $e' \notin -\Gamma e$. Then it follows that $\Lambda(v)^+ \subseteq \mathcal{T}_1^{+,\Gamma\text{-st}}$ and any two distinct elements of $\Lambda(v)^+$ are not Γ-equivalent. Thus we can find a complete set of representatives Λ_1 of $\Gamma \backslash \mathcal{T}_1^{+,\Gamma\text{-st}}$ satisfying $\Lambda(v)^+ \subseteq \Lambda_1$.

Note that for any $e \in \Lambda_1 \backslash \Lambda(v)^+$, we have $v \notin \{o(e), t(e)\}$. Hence

$$[v] \otimes \omega_{c,v} = \sum_{e \in \Lambda(v)} \varepsilon_e [v] \otimes c(\varepsilon_e e) = [v] \otimes \sum_{e \in \Lambda(v)} c(e)$$

and $\omega_{c,v} = \sum_{e \in \Lambda(v)} c(e)$, from which the lemma follows. □

Corollary 12.34. *Let Γ be an arithmetic subgroup of $GL_2(K)$ which is p'-torsion free and let $k \geqslant 2$. Then the $\mathbb{C}_\infty$-linear map*

$$\phi_\Gamma : C_k^{\mathrm{har}}(\Gamma) \to \mathrm{St} \otimes_{\mathbb{Z}[\Gamma]} V_k(\mathbb{C}_\infty), \quad c \mapsto \sum_{e \in \Lambda_1} [e] \otimes c(e)$$

is an isomorphism which is independent of the choice of Λ_1.

Proof. Since $\Lambda_1 \subseteq \mathcal{T}_1^{o,\Gamma\text{-st}}$, we have

$$\phi_\Gamma(c) = \phi_\Gamma^{\text{st}}(c|_{\mathcal{T}_1^{o,\Gamma\text{-st}}}).$$

Thus the corollary follows from Lemma 12.29 and Lemma 12.33. $\square$

Remark 12.35. The assumption that Γ is p'-torsion free in Corollary 12.34 is removed by [BGP, Theorem 1.14].

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